

Grassmannian Approach to Super-KP Hierarchies

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Abstract

We present a theory of 'maximal' super-KP(SKP) hierarchy whose flows are maximally extended to include all those of known SKP hierarchies, including, for example, the MRSKP hierarchy of Manin and Radul and the Jacobian SKP(JSKP) introduced by Mulase and Rabin. It is shown that SKP hierarchies has a natural field theoretic description in terms of the B-C system, in analogous way as the ordinary KP hierarchy. For this SKP hierarchy, we construct the vertex operators by using Kac-van de Leur superbosonization. The vertex operators act on the τ -function and then produce the wave function and the dual wave function of the hierarchy. Thereby we achieve the description of the 'maximal' SKP hierarchy in terms of the τ -function, which seemed to be lacking till now. Mutual relations among the SKP hierarchies are clarified. The MRSKP and the JSKP hierarchies are obtained as special cases when the time variables are appropriately restricted.

1 Introduction

Nearly a decade has passed since Manin and Radul introduced a supersymmetric extension of the KP hierarchy (SKP), (to be denoted as MRSKP)[1]. In the meanwhile, several variations of SKP hierarchies were born, Jacobian SKP (JSKP) hierarchy by Mulase and Rabin[2][3], SKP₂ hierarchy by Figueroa-O'Farrill et al.[4], etc. A lot of works have been made on;

- integrability and unique solvability of initial value problems, formulating in terms of flows on a super-Grassmannian[5][6],
- geometrical interpretation by extending the one of the ordinary KP hierarchy in which the orbits of the flows are canonically isomorphic to the Jacobian varieties of an algebraic curve[2], (In this respect JSKP hierarchy succeeds to the ordinary one, as its flows characterize the Jacobian variety of an algebraic 1|1 super curve.)
- construction of algebro-geometric solutions[7][8][9],
- reduction to the supersymmetric KdV(SKdV) hierarchy and its Hamiltonian structure as a completely integrable system[10].

Recent developments in understanding non-critical strings and the two-dimensional quantum gravity (coupled with various conformal fields) reveal surprising link between these theories and integrable hierarchies of the KP type[11]. It is natural to ask whether these results can be extended to the supersymmetric ground. A naive but not well founded speculation is that some supersymmetric hierarchies might underlie the two-dimensional supergravities. Though there are some proposals[12], we have not yet a definite answer. These subjects add to our interest for the SKP hierarchies. In this respect, continued from the above list,

- additional symmetries and the relation with super W -algebras

investigated by several authors[13]. However with these studies, it seems that some important elements are still missing in the theory of the SKP hierarchies. Indeed the most conspicuously lacking is the concept of the τ -function which is a vital element of the theory and this is related to the non-uniqueness of the SKP hierarchies.

Let us recall the Grassmannian approach to the KP hierarchy and the notion of the τ -function[14]. The KP hierarchy was formulated as the eigen value preserving deformations of a first order pseudo-differential operator (i.e., Lax operator) $L(t_j) = \partial_x + u_{-1}(t_j)\partial_x^{-1} + u_{-2}(t_j)\partial_x^{-2} + \dots$:

$$\begin{aligned} L(t_j) w(x, t_j; z) &= z w(x, t_j; z), \\ \partial_{t_n} L(t_j) &= (L(t_j)^n)_+ L(t_j), \quad \text{with } t_1 = x. \end{aligned}$$

Without loss of generality L is expressed as a dressed operator of ∂_x by a wave operator $S = 1 + s_{-1}\partial_x^{-1} + s_{-2}\partial_x^{-2} + \dots$, i.e. $L = S\partial_x S^{-1}$. Then the wave function w is given by the formula $w(x, t_j; z) = S(t_j) \exp \sum_{n=1} t_n z^n$. The time evolution of L is then converted to that of S which is governed by the Sato equations:

$$\partial_{t_n} S = -(S\partial_x^n S^{-1})_- S, \quad n = 1, 2, \dots$$

The Grassmannian nature of the KP hierarchy arises from the Sato correspondence which says there is a one to one correspondence between the space of the wave operators and a certain infinite dimensional Grassmannian. As a result the KP hierarchy is interpreted as a dynamical system on this Grassmannian, which is also considered as the solution space of the KP hierarchy, by viewing it as the set of initial data of the time evolution. Then $GL(\infty)$ arises as a hidden symmetry, that is the transformation group of this solution space. These things look apparent when one takes the free fermion description of the KP hierarchy, in which the Grassmannian is projectively embedded into the fermion Fock space and the KP flows are generated by the positive modes of the current operator j_n . Given an initial point represented by a state $|G\rangle$ (which sits on the $GL(\infty)$ orbit of the vacuum $|0\rangle$), an orbit of the flow is represented by $\exp \sum_{n=1} t_n j_n |G\rangle$. The contraction with the vacuum state gives the τ -function $\tau(t_j) = \langle 0 | \exp \sum_{n=1} t_n j_n |G\rangle$. Then τ maps $|G\rangle$ to a polynomial in the time variables, which is just the boson-fermion correspondence. All the information about the state $|G\rangle$ is encoded into the function $\tau(t_j)$. The locus of the Grassmannian in the fermion Fock space is characterized by the bilinear identity

$$\oint dz \psi(z)|G\rangle \otimes \psi^*(z)|G\rangle = 0,$$

which turns out to be the Hirota's bilinear equations for the τ -function. It is a great insight of Sato that reveals the Hirota's bilinear equations to be nothing but the Plücker relations. The wave function is then identified as

$$\begin{aligned} w(x, t_j; z) &= \frac{1}{\tau(t_j)} \langle 0 | \psi_{\frac{1}{2}}^* e^{\sum_{n=1} t_n j_n} \psi(z) |G\rangle \\ &= \frac{1}{\tau(t_j)} e^{\sum t_n z^n} e^{-\sum \frac{z^{-n}}{n} \partial_{t_n}} \tau(t_j), \end{aligned}$$

where the vertex operator representation of the fermion field $\psi(z)$ is used. In this way a solution for the original wave operator $S(t_j)$ is obtained from the τ -function passing through the wave function.

Turning to the SKP hierarchies, we can proceed in a parallel way. The SKP hierarchies are interpreted as the dynamical systems on the super-Grassmannian[6]. Just like the KP hierarchy allows the free fermion description, they can be fairly well formulated in a field theoretic terms employing the so called the B-C system, which is given by a tensor product of the first order system of free fermion (i.e. b-c system) and the first

order system of free boson (i.e. β - γ system). Although we have not been able to find the literatures which state this point explicitly by showing the expression for the wave function* it seems rather evident if one takes into account the following facts:

- Infinite dimensional super-Grassmannian is realized in the Fock space of the B-C system as the $GL(\infty|\infty)$ orbit of the vacuum state[16].
- It is possible to characterize, through the bilinear identity, the wave function and its dual for the SKP hierarchies[6].
- The $GL(\infty|\infty)$ orbit of the vacuum state of the B-C system satisfies the bilinear identity[17]

$$\oint dz d\theta C(z, \theta) G|0\rangle \otimes B(z, \theta) G|0\rangle = 0.$$

One of the purposes of this paper is to establish the field theoretic description of the SKP hierarchies. Instead of employing the bilinear identity mentioned above, we will follow another route to the goal. Namely we will translate the frame matrix description of the super-Grassmannian given in [6] into the language of the B-C system. At first this route might seem somewhat roundabout but it has the merit of being explicit and concrete. When one tries to construct the super-analogue of the τ -function in the SKP hierarchies, he immediately encounters a problem of (non-matching) time variables. It seems the time variables which parameterize the flows do not match any (super-)bosonization scheme of the B-C system. For example consider the well known bosonization of the β - γ system in Ref.[18]. The current of the additional fermionic pair of the b-c type usually denoted by (ξ, η) , which is necessary for the bosonization, can not be expressed as a bilinear form of the fields β and γ . Consequently the oscillators of this current can not be identified with the time evolution operators on the super-Grassmannian. Moreover simple counting of the freedom indicates the number of the time variables of the MRSKP or JSKP hierarchy is just half of the freedom necessary for (super-)bosonizing the B-C system. These observation indicates the necessity of a new SKP hierarchy. We recognize that if one likes to formulate the SKP hierarchy as a dynamical system on the super-Grassmannian, the time evolution operators should be constructed from currents in the bilinear form of the fields B and C . The possible combinations of such currents are given by $j^+ = -bc$, $j^- = -\beta\gamma$, $\psi^+ = c\beta$ and $\psi^- = \gamma b$, which form a set of generators of the Lie superalgebra $\widehat{gl(1|1)}$. Actually one finds that the flows of the MRSKP (JSKP) hierarchy are generated by the positive frequency part only of the currents $j^{tot} = -(bc + \beta\gamma)$ and $\psi = c\beta - \gamma b$ (j^{tot} and ψ^+). A superbosonization of the B-C system (the super-Weyl algebra in their terminology) based on the the Lie superalgebra $\widehat{gl(1|1)}$ has been found by Kac and van de Leur[17]. This enable us to consider a maximal SKP hierarchy

* In ref.[15], the SKP hierarchies are studied in terms of the B-C system, however the description there is not fully developed.

whose time flows are governed by all the above $\widehat{gl(1|1)}$ currents, j^\pm and ψ^\pm . In this way the MRSKP and JSKP hierarchies are naturally extended. It should be mentioned that Rabin also considered this type of a 'maximal' SKP hierarchy by connecting MRSKP and JSKP to the superbosonization of Kac-van de Leur. However, *various ways appropriately parameterizing the extended flows without spoiling the integrability* do not seem to be considered seriously. Our proposal for the time evolution operator compatible with integrability is given by (4.51). It seems to give the simplest coordinatization of the extended flows. The resulting SKP hierarchy is shown in (3.31) in the form of the Sato equations. Then we can proceed as in the case of the KP hierarchy. The τ -function can be defined, from which the wave function and its dual are derived acting the appropriate vertex operators corresponding to the field C and B . In the viewpoint of the superbosonization, we have realized (the fixed total charge sector of) the B-C system on the graded polynomial ring $\mathbf{C}[t_1^\pm, t_2^\pm, \dots]$ (where t_{2n}^\pm 's are even and t_{2n+1}^\pm 's are odd variables), although our vertex operators are somewhat complicated. In other words *we have constructed the maximal SKP hierarchy which unifies the known SKP hierarchies* at the cost of the simplicity of the corresponding superbosonization. What extent of this construction of the SKP hierarchy is considered to be natural is a remaining question.

We have succeeded in the first goal of the program for the Grassmannian approach to the SKP hierarchy. There are many subjects left for future investigations. One can think of, for example, the problems listed at the beginning, regarding the maximal SKP hierarchy. We have said virtually nothing about the geometrical aspects of the theory. We expect, however, that the field theoretic description given in this paper will shed light on our geometrical understanding of the SKP hierarchies, combining our knowledge of the conformal field theories on general (super-)Riemann surfaces[19]. (We will make some comments in the last section.)

The article organized as follows. In § 2, arranging the materials necessary for the later use, we deal with the super-Sato correspondence[2] between the space of the wave (super-pseudodifferential) operators and a certain super-Grassmannian, along the arguments in Refs.[2] and [6]. In § 3, we describe of the known SKP hierarchies in the form of the Sato equation. Then the interpretation of the hierarchies as dynamical systems on the super-Grassmannian becomes clear. With these preliminaries we present the maximal SKP hierarchy. The notion of the wave function and its dual wave function are introduced and their characterization through the bilinear identity is given in a general form. In § 4, we introduce the B-C system. Translating the frame matrix representation of the super-Grassmannian into the language of the B-C system we express the basic ingredients, i.e., the wave function, the dual wave function and the τ -function, in terms of them. We see the free fermion description of the KP hierarchy can be generalized in a very natural way to the super case. In § 5, We briefly review the superbosonization of the Kac and van de Leur and then, using this as an intermediate step, construct the vertex operators corresponding to the fields B and C . In the last section, we mention the geometrical side of the theory together with a remark on another SKP hierarchy given by LeClair[20].

2 Super Sato correspondence

To describe the SKP hierarchies as a dynamical system on the super-Grassmannian manifold, we first recall the super-Sato correspondence in some detail. Our exposition here is mostly due to [2] and [6]. Let us start with the description of the basic ingredients of the super-pseudo differential operators(SΨDO's). For more mathematical details see [2]. Let \mathcal{A} be some superalgebra modeled over \mathbf{C} . (We do not specify its precise content. The simplest case is that \mathcal{A} is \mathbf{C} itself.) We define as our function space the supercommutative algebra

$$R = \mathcal{A} \otimes_{\mathbf{C}} \mathbf{C}[[x, \xi]]. \quad (2.1)$$

Here $\mathbf{C}[[x, \xi]]$ is the ring of formal power series in even variable x and odd nilpotent variable ξ that satisfy $x\xi = \xi x$ and $\xi^2 = 0$. The \mathbf{Z}_2 gradation is introduced naturally in R from those of \mathcal{A} and $\mathbf{C}[[x, \xi]]$: $R = R_0 \oplus R_1$. R has a super-derivation operator $D = \partial_\xi + \xi \partial_x$ satisfying $D^2 = \partial_x$. We define the algebra \mathcal{E} of SΨDO's by

$$\mathcal{E} = R((D^{-j})) = \{ P = \sum_{j < \infty} p_j D^j, \quad p_j \in R \}, \quad (2.2)$$

where $D^{-1} = \partial_x^{-1} D$. Let \mathcal{D} be the subalgebra consisting of super-differential operators:

$$\mathcal{D} = R[D] = \{ P = \sum_{0 \leq j < \infty} p_j D^j \} \subset \mathcal{E}. \quad (2.3)$$

Then, we have the splitting of \mathcal{E} :

$$\mathcal{E} = \mathcal{D} \oplus \mathcal{E}_-, \quad (2.4)$$

where

$$\mathcal{E}_- = D^{-1} R[[D^{-1}]] = \{ P = \sum_{j \geq 1} p_{-j} D^{-j} \}. \quad (2.5)$$

The algebraic structure of \mathcal{E} is introduced through the super-Leibniz rule. Let f be a homogeneous element (i.e. with fixed Grassmann parity) of R . Then we have

$$D^m f = \sum_{r=0}^m \begin{bmatrix} m \\ r \end{bmatrix} (-1)^{|f|(m-r)} f^{[r]} D^{m-r}, \quad (2.6)$$

where $|f|$ denotes the Grassmann parity of f , $f^{[r]}$ denotes the r -th derivative of f with respect to D and $\begin{bmatrix} m \\ r \end{bmatrix}$ are the super-binomial coefficients[1] defined by

$$\begin{bmatrix} m \\ r \end{bmatrix} = \begin{cases} 0, & \text{for } r(m-r) = 1 \pmod{2} \\ \begin{pmatrix} \frac{m}{2} \\ \frac{r}{2} \end{pmatrix}, & \text{for } r(m-r) = 0 \pmod{2} \end{cases}. \quad (2.7)$$

As a special subalgebra of \mathcal{E} , we define Γ_0 that consists of the homogeneous even monic S Ψ DO's:

$$\Gamma_0 = \{ S \in \mathcal{E} \mid S = 1 + s_{-1}D^{-1} + s_{-2}D^{-2} + \cdots, \quad |s_j| = \begin{cases} 0, & \text{for } j \text{ even} \\ 1, & \text{for } j \text{ odd} \end{cases} \} \quad (2.8)$$

Γ_0 has a group structure (i.e super-Volterra group) since all elements of Γ_0 are invertible in Γ_0 .

Next, following Mulase[2], we give a definition of the super-Grassmannian and derive a supersymmetric generalization of the theorem of Sato, which states the relation between the S Ψ DO's and the super-Grassmannian. Let us introduce a new pair of even and odd variables z and θ which are regarded as Fourier transformations of ∂_x and ∂_ξ respectively. Furthermore it is useful to introduce symbols ζ^m , $m \in \mathbf{Z}$ [2] defined by

$$\begin{cases} \zeta^{2m} &= z^m \\ \zeta^{2m+1} &= z^m \theta. \end{cases} \quad (2.9)$$

We define the super-linear space (i.e. free \mathcal{A} -module) \mathcal{H} as the space of the formal Laurent series :

$$\mathcal{H} = \mathbf{C}((z^{-1}, \theta)) \otimes \mathcal{A} = \{ f = \sum_{n < \infty} \zeta^n f_n, \quad f_n \in \mathcal{A} \}. \quad (2.10)$$

As in the case of \mathcal{E} , the superspace structure (i.e. \mathbf{Z}_2 -gradation) of \mathcal{H} is obvious. There is a natural direct sum decomposition of \mathcal{H} :

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad (2.11)$$

where \mathcal{H}_+ and \mathcal{H}_- are the subspaces defined by respectively

$$\mathcal{H}_+ = \mathbf{C}[z, \theta] \otimes \mathcal{A} = \{ f = \sum_{0 \leq n < \infty} \zeta^n f_n \}, \quad (2.12)$$

and

$$\mathcal{H}_- = z^{-1} \mathbf{C}[[z^{-1}, \theta]] \otimes \mathcal{A} = \{ f = \sum_{n \geq 1} \zeta^{-n} f_{-n} \}. \quad (2.13)$$

Let π_+ be the projection $\pi : \mathcal{H} \longrightarrow \mathcal{H}_+$. The super-Grassmannian concerned in this paper, which is denoted by Gr_0 , is defined as a set of subspaces of \mathcal{H} such that

$$Gr_0 = \{ W \subset \mathcal{H} \mid W \oplus \mathcal{H}_- = \mathcal{H} \}. \quad (2.14)$$

In other words, the subspace $W \subset \mathcal{H}$ belongs to Gr_0 if and only if the projection of each subspace W to \mathcal{H}_+ is bijective, i.e. $\ker \pi_+|_W = 0$ and $\text{coker } \pi_+|_W = 0$. Of course this definition of the super-Grassmannian is a very restricted one corresponding to a supersymmetric generalization of the Sato Grassmannian, and see Mulase[2] for more

general definition of the super-Grassmannians with an arbitrary level and Fredholm index. Gr_0 is then viewed as a big cell of the super-Grassmannian with Fredholm index $0|0$.

The super-linear space \mathcal{H} can be viewed as an \mathcal{E} -module as follows. Having in mind the Fourier transform, we convert x , ξ , ∂_x and ∂_ξ to $-\partial_z$, ∂_θ , z and θ , and consider a homomorphism ρ from \mathcal{E} into the space of differential operators on \mathcal{H} :

$$\rho(P) = \sum_{j < \infty} \{ p_{2j}(-\partial_z, \partial_\theta) z^j + p_{2j+1}(-\partial_z, \partial_\theta) z^j (\theta + z\partial_\theta) \}, \quad P = \sum_{j < \infty} p_j(x, \xi) D^j \in \mathcal{E}. \quad (2.15)$$

Through the homomorphism ρ , SΨDO's act on \mathcal{H} and hence \mathcal{H} is considered as an \mathcal{E} -module. Furthermore, we obtain a super-linear transform from \mathcal{E} to \mathcal{H} , which is denoted by $\tilde{\rho}$, by setting

$$\tilde{\rho}(P) = \rho(P) \cdot 1 \in \mathcal{H}, \quad P \in \mathcal{E}. \quad (2.16)$$

Theorem 1 *There exists a one to one correspondence between the super-Grassmannian Gr_0 and Γ_0 . The bijection $\sigma : \Gamma_0 \xrightarrow{\sim} Gr_0$ is obtained by associating each S in Γ_0 with a subspace $\tilde{\rho}(S^{-1}\mathcal{D}) \subset \mathcal{H}$:*

$$\Gamma_0 \ni S \xrightarrow{\sigma} \tilde{\rho}(S^{-1}\mathcal{D}) \in Gr_0. \quad (2.17)$$

First we must show that the well-definedness of the map σ , i.e. $\sigma(S) \in Gr_0$. From the expression (2.15) of the homomorphism ρ we note that

1. $\tilde{\rho}(\mathcal{D}) = \mathcal{H}_+$
2. $\rho(S)\mathcal{H} = \mathcal{H}, \quad S \in \Gamma_0$
3. $\rho(S)\mathcal{H}_- = \mathcal{H}_-, \quad S \in \Gamma_0$.

Then we have $\mathcal{H} = \rho(S^{-1})\mathcal{H} = \rho(S^{-1})\mathcal{H}_+ \oplus \rho(S^{-1})\mathcal{H}_- = \tilde{\rho}(S^{-1}\mathcal{D}) \oplus \mathcal{H}_-$ for $S \in \Gamma_0$, and hence $\sigma(S) = \tilde{\rho}(S^{-1}\mathcal{D}) \in Gr_0$. To see that σ is a bijection, we need to investigate the homomorphism ρ in more detail. Let P be a homogeneous element of \mathcal{E} and express it as

$$P = \sum_{j < \infty} p_j(x, \xi) D^j, \quad p_j(x, \xi) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (a_{2n,j} + \xi a_{2n+1,j}), \quad |a_{m,j}| = m + j + |P|, \text{ mod } 2. \quad (2.18)$$

Then we set

$$v_P^{(m)} = \rho(P)\zeta^m = \sum_j \zeta^j F[P]_{jm}, \quad m \in \mathbf{Z} \quad (2.19)$$

and consider the coefficients $F[P]_{jm}$ as matrix elements of a $\mathbf{Z} \times \mathbf{Z}$ matrix $F[P]$. By calculating

$$\begin{aligned} v_P^{(m)} &= \rho(P)\zeta^m = \tilde{\rho}(PD^m) \\ &= \sum_{j<\infty} \sum_{n=0} \frac{1}{n!} (-\partial_z)^n (a_{2n,j} + \partial_\theta a_{2n+1,j}) \zeta^{j+m}, \end{aligned} \quad (2.20)$$

we obtain the following result:

$$F[P]_{jm} = (-1)^{j+m+|P|} \Xi[P]_{-(j+1),-(m+1)}, \quad j, m \in \mathbf{Z}, \quad (2.21)$$

with

$$\tilde{\Xi}[P]_{jm} = \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} \tilde{a}_{k,m-j+k}, \quad \text{where} \quad \begin{cases} \tilde{a}_{j,k} &= (-1)^{(j+|P|)(k+1)} a_{j,k} \\ \tilde{\Xi}[P]_{jk} &= (-1)^{(j+|P|)(k+1)} \Xi[P]_{jk}. \end{cases} \quad (2.22)$$

We write (2.21) in the matrix form

$$F[P] = (-1)^{|P|} JK\Xi[P]KJ, \quad (2.23)$$

where J and K are the matrices defined by

$$J = ((-1)^j \delta_{jk})_{j,k \in \mathbf{Z}} \quad \text{and} \quad K = (\delta_{j+k+1,0})_{j,k \in \mathbf{Z}}. \quad (2.24)$$

Using the properties of the super-binomial coefficients, the expression (2.22) can be inverted as

$$\tilde{a}_{j,m} = \sum_{k=0}^j \begin{bmatrix} j \\ k \end{bmatrix} (-1)^{\frac{j(j+1)-k(k+1)}{2}} \tilde{\Xi}[P]_{k,m-j+k}, \quad j \in \mathbf{Z}_{\geq}, \quad m \in \mathbf{Z}. \quad (2.25)$$

We see that for an arbitrary integer p , $(a_{j,m})_{j \in \mathbf{Z}_{\geq}, m \in p+\mathbf{Z}_{<}}$ and $(\Xi[P]_{jm})_{j \in \mathbf{Z}_{\geq}, m \in p+\mathbf{Z}_{<}}$ can be expressed each other by the above transforms.

Lemma 1 *If a SΨDO P preserves \mathcal{H}_+ , i.e. $\rho(P)\mathcal{H}_+ \subset \mathcal{H}_+$, $P \in \mathcal{E}$, then P must be a differential operator, i.e. $P \in \mathcal{D}$.*

[Proof] The condition $\rho(P)\mathcal{H}_+ \subset \mathcal{H}_+$ means $v_P^{(m)} \in \mathcal{H}_+$, $m = 0, 1, 2, \dots$ and so $F[P]_{jm} = 0$, for $j \in \mathbf{Z}_{<}$, $m \in \mathbf{Z}_{\geq}$, (equivalently $\Xi[P]_{jm} = 0$, for $j \in \mathbf{Z}_{\geq}$, $m \in \mathbf{Z}_{<}$). Then it follows from (2.25) that $a_{jm} = 0$, for $j \in \mathbf{Z}_{\geq}$, $m \in \mathbf{Z}_{<}$, showing $P \in \mathcal{D}$. \square

Let $P \in \Gamma_0$ then in (2.15)

$$\begin{aligned} a_{jm} &= 0, & \text{for } m > 0, \\ a_{j0} &= 0, & \text{for } j > 0, \text{ and } a_{00} = 1. \end{aligned} \quad (2.26)$$

In terms of $\Xi[P]$ we find

1. $\Xi[P]_{jm} = \delta_{jm}$ for $j \leq m$ (i.e. $\Xi[P]$ becomes a triangular matrix with unit diagonal elements.)
2. In specifying $P \in \Gamma_0$ we are free to set the values $(\Xi[P]_{jm})_{j \in \mathbf{Z}_{\geq}, m \in \mathbf{Z}_{<}}$ (equivalently $(a_{j,m})_{j \in \mathbf{Z}_{\geq}, m \in \mathbf{Z}_{<}}$). The other elements of the triangular matrix $\Xi[P]$ are determined by that part of the elements as

$$\Xi_{jm} = \sum_{k=0}^{j-m} f(j, m)_k \Xi_{k, k-(j-m)} \quad \text{for } j \geq m > 0 \quad (2.27)$$

$$\Xi_{-(j+1), -(m+1)} = \sum_{k=0}^{m-j} h(j, m)_k \Xi_{k, k-(m-j)} \quad \text{for } m \geq j \geq 0 \quad (2.28)$$

We do not need to know the explicit forms of the coefficients $f(j, m)_k$ and $h(j, m)_k$ in the following.

Conversely, we have the following lemma.

Lemma 2 *A triangular matrix (Ξ_{ij}) with $\Xi_{ij} = \delta_{ij}$ for $j \leq m$ is associated with some $S\Psi DO S \in \Gamma_0$ if and only if it satisfies the relations (2.27) and (2.28) among its elements.*

[Proof] By (2.25), $(\Xi_{jm})_{j \in \mathbf{Z}_{\geq}, m \in \mathbf{Z}_{<}}$ determines $(a_{j,m})_{j \in \mathbf{Z}_{\geq}, m \in \mathbf{Z}_{<}}$ and so $S \in \Gamma_0$. \square

Now we return to the theorem.

(i) Injectivity of σ

Suppose that $\sigma(S_1) = \sigma(S_2)$. Then we have

$$\tilde{\rho}(S_1 S_2^{-1} \mathcal{D}) = \tilde{\rho}(\mathcal{D}) = \mathcal{H}_+.$$

From lemma 1, it follows $S_1 S_2^{-1} \in \mathcal{D} \cap \Gamma_0 = \{1\}$, thus we have $S_1 = S_2$.

(ii) Surjectivity of σ

Let W be an arbitrary element of Gr_0 . To prove the surjectivity, it is sufficient to find a basis for W of the form

$$v^{(m)}(\zeta) = \sum_{k \leq m} (-1)^{k+m} \zeta^k \Xi_{-(k+1), -(m+1)}, \quad m = 0, 1, 2, \dots \quad (2.29)$$

where the set of coefficients $(\Xi_{jk})_{j \geq k, k \in \mathbf{Z}_{<}}$ can be considered as a part of a triangular matrix that satisfies the relations (2.27) and (2.28). Then Lemma 2 indicates $\Xi = \Xi[S]$ with some $S \in \Gamma_0$ and (2.29) is nothing but $\tilde{\rho}(S\mathcal{D}) = \sigma(S^{-1}) = W$, which shows the surjectivity of σ . Let $\{w^{(m)}\}$, $m = 0, 1, 2, \dots$ be the canonical basis for w which takes the form

$$w^{(m)}(\zeta) = \zeta^m + \sum_{k > 0} \zeta^{-k} w_{-k, m}. \quad (2.30)$$

We can determine inductively the matrix elements Ξ_{jk} from the coefficients $(w_{j,k})$ to satisfy the relation (2.27) and (2.28). First we set $v^{(0)} = w^{(0)}$ which fixes -1 st column of Ξ as

$$\Xi_{-1,-1} = 1, \quad \text{and} \quad \Xi_{k,-1} = (-1)^{k+1} w_{-(k+1),0} \quad k = 0, 1, 2, \dots$$

Suppose that we have determined the matrix Ξ up to $-n$ -th column, i.e. Ξ_{jk} with $0 > k \geq -n$, $j \geq k$. Then $\Xi_{-(j+1),-(n+1)}$, $0 \leq j \leq n$ can be fixed from (2.28), and we set

$$v^{(n)}(\zeta) = w^{(n)}(\zeta) + (-1)^n \sum_{j=1} \Xi_{-(n-j+1),-(n+1)} w^{(n-j)}(\zeta),$$

which determine the elements $\Xi_{j,-(n+1)}$ $j \geq 0$. The relation (2.27) only states positive columns of Ξ should be fixed by the negative ones. Now we complete the proof of the theorem.

Let $\Xi \in \text{Mat}(\mathbf{Z} \times \mathbf{Z}, \mathcal{A})$. We have observed in (2.23) that for a given $S \in \Gamma_0$, $\Xi[S^{-1}]^<^\dagger$ defined through (2.22) gives a frame matrix for an associated subspace $W = \sigma(S) \in Gr_0$. There exists a natural way to obtain $\Xi[P]$ from a given $S\Psi\text{DO}$ $P \in \mathcal{E}$. Let us define a $\mathbf{Z} \times \mathbf{Z}$ matrix $\psi[P] = (\psi[P]_{jk})$, $P \in \mathcal{E}$ by

$$D^j P = \sum_k \psi[P]_{jk} D^k. \quad (2.32)$$

then the mapping ψ

$$\mathcal{E} \ni P \xrightarrow{\psi} \psi[P] \in \text{Mat}(\mathbf{Z} \times \mathbf{Z}, R). \quad (2.33)$$

becomes an injective algebra homomorphism and thus gives a matrix representation of the algebra \mathcal{E} . The following proposition shows how the matrix $\Xi[P]$ arises from $\psi[P]$.

Proposition 1 *For a given $S\Psi\text{DO}$ P , the matrix $\psi[P]$ takes the form*

$$\psi[P](x, \xi) = e^{x\Lambda + \xi\Gamma} \Xi[P] e^{-(x\Lambda + \xi\Gamma)}, \quad (2.34)$$

and hence $\Xi[P] = \psi[P]|_{x=\xi=0}$, where $\Xi[P]$ is the matrix defined previously, and Γ and Λ are the matrices given by

$$\Gamma = (\delta_{j+1,k})_{j,k \in \mathbf{Z}} \quad \text{and} \quad \Lambda = (\delta_{j+2,k})_{j,k \in \mathbf{Z}} = \Gamma^2. \quad (2.35)$$

[†] Throughout this paper, the subscripts \leq (\geq) and $<$ ($>$) indicate taking the part of non-negative (non-positive) integers and the part of negative (positive) integers, respectively. For example, let $\Xi \in \text{Mat}(\mathbf{Z} \times \mathbf{Z}, \mathcal{A})$, then Ξ is written in a block form

$$\Xi = \begin{pmatrix} \Xi_{\geq\geq} & \Xi_{\geq<} \\ \Xi_{<\geq} & \Xi_{<<} \end{pmatrix}.$$

We further set

$$\Xi_{\geq} = \begin{pmatrix} \Xi_{\geq\geq} \\ \Xi_{<\geq} \end{pmatrix} \quad \text{and} \quad \Xi_{<} = \begin{pmatrix} \Xi_{\geq<} \\ \Xi_{<<} \end{pmatrix}. \quad (2.31)$$

It is useful to introduce a pairing $\langle , \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{A}$ defined by

$$\langle f, g \rangle = \frac{1}{2\pi i} \oint_0 dz d\theta f(\zeta) g(\zeta), \quad f, g \in \mathcal{H} \quad (2.36)$$

\langle , \rangle has the properties

$$\begin{aligned} \langle \lambda f, g \rangle &= (-1)^{|\lambda|} \langle f, g \rangle, & \langle f, g \lambda \rangle &= \langle f, g \rangle \lambda, \\ \langle f \lambda, g \rangle &= \langle f, \lambda g \rangle, & \lambda &\in \mathcal{A}, \\ \langle \zeta^m, \zeta^n \rangle &= \delta_{n+m+1, 0}. \end{aligned}$$

To prove the proposition, let us consider the following integral

$$\frac{1}{2\pi i} \oint dz d\theta D^k e^{xz+\xi\theta} \rho(P) \zeta^m = \langle D^k e^{xz+\xi\theta}, \rho(P) \zeta^m \rangle, \quad P \in \mathcal{E}. \quad (2.37)$$

Here we note the following identities:

$$\begin{aligned} \langle D^k e^{xz+\xi\theta}, \zeta^j \rangle &= \langle e^{xz+\xi\theta}, \zeta^{k+j} \rangle \\ &= \begin{cases} \frac{x^n}{n!} & \text{for } k+j+1 = -2n, \\ \frac{-\xi x^n}{n!} & \text{for } k+j+1 = -(2n+1), \\ 0 & \text{otherwise} \end{cases} \quad \begin{matrix} n \in \mathbf{Z}_{\geq} \\ n \in \mathbf{Z}_{\geq} \end{matrix} \\ &= (e^{x\Lambda - \xi\Gamma})_{k, -(j+1)} = (e^{x\Lambda - \xi\Gamma} K)_{k, j}, \end{aligned} \quad (2.38)$$

from which it follows that

$$D^k e^{xz+\xi\theta} = \sum_j (e^{x\Lambda + \xi\Gamma})_{kj} \zeta^j. \quad (2.39)$$

In addition we see for $P \in \mathcal{E}$, $f \in \mathcal{H}$, integrating by part

$$\langle e^{xz+\xi\theta}, \rho(P) f \rangle = \langle P e^{xz+\xi\theta}, f \rangle. \quad (2.40)$$

Making use of these equalities, we can calculate the expression (2.37) in two ways. On the one hand, using the notation (2.19), we have

$$\begin{aligned} \langle D^k e^{xz+\xi\theta}, \rho(P) \zeta^m \rangle &= \sum_j \langle D^k e^{xz+\xi\theta}, \zeta^j \rangle F[P]_{jm} \\ &= (e^{x\Lambda - \xi\Gamma} K F[P])_{km}. \end{aligned}$$

On the other hand, from the definition of $\psi[P]$, we have

$$\begin{aligned} \langle D^k e^{xz+\xi\theta}, \rho(P) \zeta^m \rangle &= \langle D^k P e^{xz+\xi\theta}, \zeta^m \rangle \\ &= \sum_j (-1)^{k+j+|P|} \psi[P]_{kj} \langle D^j e^{xz+\xi\theta}, \zeta^m \rangle \\ &= (-1)^{|P|} (J \psi[P] J e^{x\Lambda - \xi\Gamma} K)_{km}. \end{aligned}$$

As a result we obtain

$$\psi[P]J e^{x\Lambda - \xi\Gamma} K = (-1)^{|P|} J e^{x\Lambda - \xi\Gamma} K F[P]. \quad (2.41)$$

Putting (2.23) into the above formula and taking into account of the matrix identities $J\Gamma + \Gamma J = \mathbf{0} = JK + KJ$ and $K^2 = J^2 = \mathbf{1}$, we get (2.34).

Lastly, we introduce the “adjoint” operation $*$ in \mathcal{E} , which plays an important role in the field theoretic representation of the SKP hierarchy, and consider the corresponding formula in the matrix representation ψ . The adjoint operation is an involution which is defined uniquely from

- (i) $D^* = -D$,
- (ii) $f^* = f$, for $f \in R$,
- (iii) $(P_1 P_2)^* = (-1)^{|P_1||P_2|} P_2^* P_1^*$ for any two homogeneous elements $P_1, P_2 \in \mathcal{E}$.

Proposition 2 *Let $P \in \mathcal{E}$, then*

$$\psi[P^*] = (-1)^{|P|} I K \psi[P]^{\text{st}} K I, \quad (2.42)$$

where I is a diagonal matrix given by

$$I = ((-1)^{\frac{j(j+1)}{2}} \delta_{jk})_{j,k \in \mathbf{Z}}, \quad (2.43)$$

and $\psi[P]^{\text{st}}$ is the supertransposition of $\psi[P]$.

The definition of the supertransposition (we adopted in this paper) is as follows. Let $X = (X_{jk})_{j,k \in \mathbf{Z}}$ be a homogeneous element of $\text{Mat}(\mathbf{Z} \times \mathbf{Z}, R)$, i.e. $|X_{jk}| = |X| + j + k \pmod{2}$. Then we set

$$(X^{\text{st}})_{jk} = (-1)^{(|X|+k)(j+k)} X_{kj}. \quad (2.44)$$

The supertransposition (2.44) is defined so as to satisfy

$$(XY)^{\text{st}} = (-1)^{|X||Y|} Y^{\text{st}} X^{\text{st}} \quad (2.45)$$

for all homogeneous $X, Y \in \text{Mat}(\mathbf{Z} \times \mathbf{Z}, R)$. Note that successive operations give

$$(X^{\text{st}})^{\text{st}} = J X J. \quad (2.46)$$

Now Proposition 2 follows from the definition of ψ . Taking the adjoint of

$$D^j P = \sum_k \psi[P]_{jk} D^k, \quad P \in \mathcal{E},$$

we get

$$(-1)^{\frac{j(j+1)}{2}+|P|j} P^* D^j = \sum_k (-1)^{\frac{k(k+1)}{2}+(|P|+k+j)k} D^k \psi[P]_{jk}.$$

Applying D^m , we further obtain

$$\begin{aligned} (-1)^{\frac{j(j+1)}{2}+|P|j} \sum_l \psi[P^*]_{ml} D^{l+j} &= \sum_k (-1)^{\frac{k(k+1)}{2}+(|P|+k+j)k} D^{m+k} \psi[P]_{jk} \\ &= \sum_k \sum_{r=0}^{m+k} (-1)^{\frac{k(k+1)}{2}+(|P|+k+j)(m-r)} \begin{bmatrix} m+k \\ r \end{bmatrix} \psi[P]_{jk}^{[r]} D^{m+k-r}. \end{aligned}$$

Comparing the coefficients on both sides, we see

$$\begin{aligned} &(-1)^{\frac{j(j+1)}{2}+|P|j} \psi[P^*]_{ml} \\ &= \sum_{r=0} (-1)^{\frac{(l+j+r-m)(l+j+r-m+1)}{2}+(|P|+l+r-m)(m-r)} \begin{bmatrix} l+j+r \\ r \end{bmatrix} \psi[P]_{j,l+j+r-m}^{[r]}. \end{aligned}$$

Putting $j = -(l+1)$ and taking account of $\begin{bmatrix} r-1 \\ r \end{bmatrix} = \delta_{r,0}$, we get

$$\psi[P^*]_{ml} = (-1)^{\frac{(l-m)(l-m+1)}{2}+|P|(m+l+1)} \psi[P]_{-(l+1),-(m+1)}, \quad (2.47)$$

which gives the matrix identity (2.42).

The following facts will be useful later on.

Proposition 3 *Let $P, Q \in \mathcal{E}$, then*

$$(i) \quad P^* e^{-(xz+\xi\theta)} = \rho(P) e^{-(xz+\xi\theta)}. \quad (2.48)$$

(ii) $P \in \mathcal{D}$ if and only if

$$\langle e^{xz+\xi\theta}, \rho(P) \zeta^m \rangle = 0, \quad m = 0, 1, 2, \dots,$$

or equivalently

$$\langle e^{xz+\xi\theta}, \rho(P) e^{-(x'z+\xi'\theta)} \rangle = 0. \quad (2.49)$$

(iii) $PQ^* \in \mathcal{D}$ if and only if

$$\langle P(x, \xi) e^{xz+\xi\theta}, Q(x', \xi') e^{-(x'z+\xi'\theta)} \rangle = 0. \quad (2.50)$$

[Proof] Let $P = \sum_{j<\infty} p_j(x, \xi) D^j$. (i) follows from a straightforward computation.

$$\begin{aligned} P^* e^{-(xz+\xi\theta)} &= \sum_j (-1)^{\frac{j(j+1)}{2}+(j+|P|)j} D^j p_j(x, \xi) e^{-(xz+\xi\theta)} \\ &= \sum_j (-1)^{\frac{j(j+1)}{2}+(j+|P|)j} D^j p_j(-\partial_z, \partial_\theta) e^{-(xz+\xi\theta)} \end{aligned}$$

$$\begin{aligned}
&= \sum_j (-1)^{\frac{j(j+1)}{2}} p_j (-\partial_z, \partial_\theta) D^j e^{-(xz+\xi\theta)} \\
&= \sum_j p_j (-\partial_z, \partial_\theta) (\theta + z\partial_\theta)^j e^{-(xz+\xi\theta)} \\
&= \rho(P) e^{-(xz+\xi\theta)}.
\end{aligned} \tag{2.51}$$

(ii) is simply a paraphrase of Lemma 1 since for $f(\zeta) \in \mathcal{H}$, $\langle e^{xz+\xi\theta}, f(\zeta) \rangle = 0$ means $f(\zeta) \in \mathcal{H}_+$ and hence (2.50) is a statement $\rho(P)\mathcal{H}_+ \subset \mathcal{H}_+$.

(iii) follows from (i) and (ii). \square

3 Time evolution

3.1 Time evolution and the known SKP hierarchies

The super-Sato correspondence leads us to the natural interpretation of SKP hierarchies as dynamical system on the super-Grassmannian Gr_0 . As stated in Introduction, various SKP hierarchies emerge according to various ways of introducing the set of infinitely many (even and odd) flows on the Gr_0 . First we start to describe the two known SKP hierarchies, i.e. MRSKP and JSKP hierarchies. Let $\{t_n\}_{n \geq 1}$ be the set of an infinite number of the time variables, where the even times $\{t_{2n}\}_{n \geq 1}$ are even variables and odd times $\{t_{2n-1}\}_{n \geq 1}$ are odd variables, respectively. Now we define the time evolution operators U_{MR} and U_J

$$\begin{aligned}
U_{MR} &= e^{H_{MR}}, & \text{with} & \quad H_{MR} = \sum_{n=1} t_n D^n, \\
U_J &= e^{H_J}, & \text{with} & \quad H_J = \sum_{n=0} (t_{2(n+1)} \partial_x^{n+1} + t_{2n+1} \partial_x^n \partial_\xi).
\end{aligned} \tag{3.1}$$

U_{MR} and U_J act on Gr_0 through $\rho(U_{MR}, U_J)Gr_0$ and define the flows for MRSKP and JSKP hierarchies. In introducing the infinitely many time variables, we should extend the super algebra \mathcal{A} to $\mathcal{A}[[t_1, t_2, \dots]]$ and also extend the definition of \mathcal{E} , \mathcal{E}_- , \mathcal{D} and Γ_0 appropriately to accommodate the time dependence. (See, for example, the prescription in reference [2].) Here, we abuse the same notation \mathcal{E} , \mathcal{E}_- , \mathcal{D} and Γ_0 allowing them to have the time dependence. Assume $U(t) = e^{H(t)}$ (with $H = H_{MR}$ or H_J) generates the flows on Gr_0 . It means that, for an arbitrary element $W = \rho(S_0^{-1})\mathcal{H}_+$ we have $\rho(U(t))W = \rho(U(t)S_0^{-1})\mathcal{H}_+ \in Gr_0$. Then, from the theorem in the previous section, there exists a SΨDO $S(t) \in \Gamma_0$ such that

$$\rho(U(t)S_0^{-1})\mathcal{H}_+ = \rho(S^{-1}(t))\mathcal{H}_+. \tag{3.2}$$

Thus $S(t)U(t)S_0^{-1}$ preserves \mathcal{H}_+ , and hence from Lemma 1 we have

$$(S(t)U(t)S_0^{-1})_- = 0. \tag{3.3}$$

We refer Eq.(3.3) as the Grassmann equation according to [6]. In [5] Mulase established the supersymmetric generalization of the Birkhoff decomposition, which states the unique solvability of Eq.(3.3), in other words, $U(t)S_0^{-1}$ is factorized uniquely into the form

$$U(t)S_0^{-1} = S(t)^{-1}Y(t), \quad S(t) \in \Gamma_0, \quad Y(t) \in \mathcal{D} \quad \text{s.t.} \quad Y(0) = 1. \quad (3.4)$$

Note that the differential operator $Y(t)$ is invertible in \mathcal{D} . The operator $S(t)$ satisfies the system of the equations (i.e. the SKP hierarchy) derived from (3.3). Let us define a one form

$$\Omega \equiv dU(t)U(t)^{-1} = \sum_{n=1} dt_n \Omega_n, \quad (3.5)$$

where $d = \sum_{n=1} dt_n \partial_{t_n}$. The explicit forms of the operators Ω_n are

$$\begin{cases} \Omega_{2n} &= \partial_x^n, \\ \Omega_{2n+1} &= \partial_x^n \partial_\xi. \end{cases} \quad \text{for JSKP} \quad (3.6)$$

$$\begin{cases} \Omega_{2n} &= \partial_x^n, \\ \Omega_{2n+1} &= D^{2n+1}(1 - \sum_{m=0} t_{2m+1} D^{2m+1}), \end{cases} \quad \text{for MRSKP}$$

Applying d to (3.4) and decomposing it into the \mathcal{E}_- part and \mathcal{D} part, we have

$$dS = -(S\Omega S^{-1})_- S, \quad (3.7)$$

$$dY = (S\Omega S^{-1})_+ Y. \quad (3.8)$$

Equation (3.7) is equivalent to the system

$$\begin{cases} \partial_{t_{2n}} S &= -(S\partial_x^n S^{-1})_- S, \\ \partial_{t_{2n+1}} S &= -(S\partial_x^n \partial_\xi S^{-1})_- S, \end{cases} \quad \text{for JSKP} \quad (3.9)$$

and

$$D_n S = -(SD^n S^{-1})_- S, \quad \text{for MRSKP} \quad (3.10)$$

where

$$\begin{cases} D_{2n} &= \partial_{t_{2n}}, \\ D_{2n+1} &= \partial_{t_{2n+1}} - \sum_{m=0} t_{2m+1} \partial_{t_{2(n+m+1)}}. \end{cases} \quad (3.11)$$

Here we mention the unique solvability of the initial value problem for the system (3.9) and (3.10) proved by Mulase in [5]. The argument is outlined as follows. To obtain a unique solution of (3.7), first solve

$$dZ = \Omega Z, \quad (3.12)$$

where Z is a SΨDO such that $Z(0) \in \Gamma_0$. The generalized Birkhoff decomposition theorem previously mentioned gives unique factorization $Z(t) = S(t)^{-1}Y(t)$, $S(t) \in \Gamma_0$, $Y(t) \in \mathcal{D}$ s.t. $Y(0) = 1$. Then $S(t)$ and $Y(t)$ solve (3.7) and (3.8) respectively, as viewed in the previous paragraph. The uniqueness of the solution $S(t)$ results from the unique solvability (of the initial value problem) of the equation (3.12) (at least when we restrict ourselves $\mathcal{A} = \mathbf{C}$) and the uniqueness of the factorization of Z .

3.2 Maximal SKP hierarchy

Looking closely at the form U_J and U_{MR} , we now present a new time evolution operator which generates all the flows of JSKP and MRSKP hierarchies. This corresponds to the one introduced as ‘maximal’ SKP hierarchy in [3], so we will call the resultant system by this name. In order to realise the ‘maximal’ SKP hierarchy, we have to double the number of time variables, i.e. we introduce two infinite sets of the time variables $\{t_n^+\}_{n \geq 1}$ and $\{t_n^-\}_{n \geq 1}$, where even times t_{2n}^\pm and odd times t_{2n+1}^\pm are even and odd respectively as before. The time evolution operator we consider is defined as

$$U_M = U_{(1)}U_{(0)}, \quad (3.13)$$

where

$$U_{(1)} = e^{H_{(1)}} \quad \text{with} \quad H_{(1)} = \sum_{n=0} (t_{2n+1}^+ \partial_\xi \partial_x^n + t_{2n+1}^- \xi \partial_x^{n+1}), \quad (3.14)$$

$$U_{(0)} = e^{H_{(0)}} \quad \text{with} \quad H_{(0)} = \sum_{n=0} (t_{2(n+1)}^+ \partial_\xi \xi \partial_x^{n+1} + t_{2(n+1)}^- \xi \partial_\xi \partial_x^{n+1}). \quad (3.15)$$

Since $H_{(1)}$ and $H_{(0)}$ do not commute each other, exponentiation of all the vector fields $H_{(1)}$ and $H_{(0)}$ in a form such as $e^{H_{(1)}+H_{(0)}}$ turns out unsuccessful. Instead we consider successive time evolutions: $U_{(0)}$ first generates even flows and subsequently $U_{(1)}$ generates odd flows. From the definition, it is evident that the JSKP and the MRSKP hierarchies are special cases of this maximal hierarchy and are obtained by setting the time variables respectively,

$$\begin{aligned} t_{2n}^+ &= t_{2n}^- = t_{2n}, & t_{2n+1}^- &= 0, & t_{2n+1}^+ &\rightarrow t_{2n+1} & \text{for JSKP,} \\ t_{2n}^+ &= t_{2n}^- = t_{2n}, & t_{2n+1}^+ &= t_{2n+1}^- = t_{2n+1} & & \text{for MRSKP.} \end{aligned} \quad (3.16)$$

As before we define the one form

$$\Omega_M = \sum_{n=1} (dt_n^+ \Omega_n^+ + dt_n^- \Omega_n^-) \quad (3.17)$$

by

$$\Omega_M \equiv dU_M U_M^{-1} = dU_{(1)} U_{(1)}^{-1} + U_{(1)} (dU_{(0)} U_{(0)}^{-1}) U_{(1)}^{-1}, \quad (3.18)$$

where $d = \sum_{n=1} (dt_n^+ \partial_{t_n^+} + dt_n^- \partial_{t_n^-})$. Then we have

$$\begin{aligned} \Omega_{2n+1}^+ &= \partial_\xi \partial_x^n + \frac{1}{2} \sum_{m=0} t_{2m+1}^- \partial_x^{n+m+1}, \\ \Omega_{2n+1}^- &= \xi \partial_x^{n+1} + \frac{1}{2} \sum_{m=0} t_{2m+1}^+ \partial_x^{n+m+1}, \end{aligned} \quad (3.19)$$

$$\begin{aligned} \Omega_{2n}^+ &= \partial_\xi \xi \partial_x^n - \sum_{m=0} (t_{2m+1}^+ \Omega_{2(n+m)+1}^+ - t_{2m+1}^- \Omega_{2(n+m)+1}^-), \\ \Omega_{2n}^- &= \xi \partial_\xi \partial_x^n + \sum_{m=0} (t_{2m+1}^+ \Omega_{2(n+m)+1}^+ - t_{2m+1}^- \Omega_{2(n+m)+1}^-). \end{aligned} \quad (3.20)$$

The calculation of Ω_M is carried out as follows. First we note $H_{(1)}^3 = 0$, which follows from the anti-commutativity of the odd times among themselves. We have thus

$$U_{(1)} = 1 + H_{(1)} + \frac{1}{2}H_{(1)}^2, \quad \text{and} \quad U_{(1)}^{-1} = 1 - H_{(1)} + \frac{1}{2}H_{(1)}^2, \quad (3.21)$$

Similarly, noting $H_{(1)}dH_{(1)}H_{(1)} = 0$, we have

$$\begin{aligned} \Omega_{(1)} &\equiv dU_{(1)}U_{(1)}^{-1} \\ &= \left\{ dH_{(1)} + \frac{1}{2} \left(dH_{(1)}H_{(1)} + H_{(1)}dH_{(1)} \right) \right\} \left(1 - H_{(1)} + \frac{1}{2}H_{(1)}^2 \right) \\ &= dH_{(1)} + \frac{1}{2}[H_{(1)}, dH_{(1)}]. \end{aligned} \quad (3.22)$$

Putting into (3.22) the expression (3.14) of $H_{(1)}$, we obtain (3.19). Next let us consider the even time part of Ω . It is convenient to separate $H_{(0)}$ and $H_{(1)}$ respectively into two terms so that the each term contains only one type of time variables (i.e. t_n^+ or t_n^-):

$$H_{(1)} = H_{(1)}^+ + H_{(1)}^- \quad \text{with} \quad \begin{cases} H_{(1)}^+ = \sum_{n=0} t_{2n+1}^+ \partial_\xi \partial_x^n, \\ H_{(1)}^- = \sum_{n=0} t_{2n+1}^- \xi \partial_x^{n+1}, \end{cases} \quad (3.23)$$

$$H_{(0)} = H_{(0)}^+ + H_{(0)}^- \quad \text{with} \quad \begin{cases} H_{(0)}^+ = \sum_{n=1} t_{2n}^+ \partial_\xi \xi \partial_x^n, \\ H_{(0)}^- = \sum_{n=1} t_{2n}^- \xi \partial_\xi \partial_x^n. \end{cases} \quad (3.24)$$

Because of $H_{(0)}^+ H_{(0)}^- = H_{(0)}^- H_{(0)}^+ = 0$, we have

$$U_{(0)} = e^{H_{(0)}^+ + H_{(0)}^-} = e^{H_{(0)}^+} + e^{H_{(0)}^-} - 1 \quad (3.25)$$

and

$$\begin{aligned} dU_{(0)}U_{(0)}^{-1} &= \left(dH_{(0)}^+ e^{H_{(0)}^+} + dH_{(0)}^- e^{H_{(0)}^-} \right) \left(e^{-H_{(0)}^+} + e^{-H_{(0)}^-} - 1 \right) \\ &= dH_{(0)} \end{aligned} \quad (3.26)$$

Furthermore let us write $H_{(0)}$ as

$$H_{(0)} = E + \bar{E} \quad \text{with} \quad \begin{cases} E = \sum_{n=1} t_{2n} \partial_x^n, \\ \bar{E} = \sum_{n=1} \bar{t}_{2n} (\partial_\xi \xi - \xi \partial_\xi) \partial_x^n, \end{cases} \quad (3.27)$$

where t_{2n} and \bar{t}_{2n} are defined as

$$t_{2n} = \frac{t_{2n}^+ + t_{2n}^-}{2} \quad \text{and} \quad \bar{t}_{2n} = \frac{t_{2n}^+ - t_{2n}^-}{2}. \quad (3.28)$$

Noting that $H_{(1)}E = EH_{(1)}$ and $H_{(1)}\bar{E} = -\bar{E}H_{(1)}$, we have

$$\begin{aligned}
\Omega_{(0)} &\equiv U_{(1)} \left(dU_{(0)} U_{(0)}^{-1} \right) U_{(1)}^{-1} \\
&= \left(1 + H_{(1)} + \frac{1}{2} H_{(1)}^2 \right) (dE + d\bar{E}) \left(1 - H_{(1)} + \frac{1}{2} H_{(1)}^2 \right) \\
&= dE + d\bar{E} \left(1 - 2H_{(1)} + 2H_{(1)}^2 \right).
\end{aligned} \tag{3.29}$$

Taking into account the expressions \bar{E} and $H_{(1)}^\pm$, we observe that

$$(\partial_\xi \xi - \xi \partial_\xi) H_{(1)}^\pm = \pm H_{(1)}^\pm, \quad (\partial_\xi \xi - \xi \partial_\xi) H_{(1)}^2 = [H_{(1)}^+, H_{(1)}^-]$$

and hence

$$\begin{aligned}
(\partial_\xi \xi - \xi \partial_\xi) (H_{(1)} - H_{(1)}^2) &= H_{(1)}^+ - H_{(1)}^- - [H_{(1)}^+, H_{(1)}^-] \\
&= \sum_{n=0} \left\{ t_{2n+1}^+ \left(\partial_\xi + \frac{1}{2} \sum_{m=0} t_{2m+1}^- \partial_x^{n+m+1} \right) - (t^+ \leftrightarrow t^-) \right\} \\
&= \sum_{n=0} (t_{2n+1}^+ \Omega_{2n+1}^+ - t_{2n+1}^- \Omega_{2n+1}^-).
\end{aligned} \tag{3.30}$$

Making use of the above equalities, we obtain the expressions (3.20) from (3.29). As in the previous case, assuming that Gr_0 is preserved through the time evolution raised by U_M , we are led to Eq.(3.3). Then the SΨDO $S(t)$ satisfies Eq.(3.7) with Ω_M . From (3.19) and (3.20), Eq.(3.7) reads

$$\begin{aligned}
D_{2n+1}^+ S &= -(S \partial_\xi \partial_x^n S^{-1})_- S, \\
D_{2n+1}^- S &= -(S \xi \partial_x^{n+1} S^{-1})_- S, \\
D_{2n}^+ S &= -(S \partial_\xi \xi \partial_x^n S^{-1})_- S, \\
D_{2n}^- S &= -(S \xi \partial_\xi \partial_x^n S^{-1})_- S,
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
D_{2n+1}^\pm &= \partial_{t_{2n+1}^\pm} - \frac{1}{2} \sum_{m=0} t_{2m+1}^\mp (\partial_{t_{2(n+m+1)}^+} + \partial_{t_{2(n+m+1)}^-}) \\
&= \partial_{t_{2n+1}^\pm} - \frac{1}{2} \sum_{m=0} t_{2m+1}^\mp \partial_{t_{2(n+m+1)}},
\end{aligned} \tag{3.32}$$

$$D_{2n}^\pm = \partial_{t_{2n}^\pm} \pm \sum_{m=0} (t_{2m+1}^+ \partial_{t_{2(n+m+1)}^+} - t_{2m+1}^- \partial_{t_{2(n+m+1)}^-}). \tag{3.33}$$

The above system of equations defines the maximal SKP hierarchy. It includes all the flows of MRSKP and JSKP as subflows. This is first main result of the present paper.

Note that $D_{2n}^+ + D_{2n}^- = \partial_{t_{2n}^+} + \partial_{t_{2n}^-} = \partial_{t_{2n}}$ with t_{2n} being defined in (3.28). The time differential operators D_n^\pm , ($n \geq 1$) satisfy the commutation relations

$$\begin{aligned} \{D_{2m+1}^+, D_{2n+1}^-\} &= -(D_{2(m+n+1)}^+ + D_{2(m+n+1)}^-) = -\partial_{t_{2(m+n+1)}}, \\ \{D_{2m+1}^+, D_{2n+1}^+\} &= \{D_{2m+1}^-, D_{2n+1}^-\} = 0, \\ [D_{2m+1}^+, D_{2n}^\pm] &= \pm D_{2(n+m)+1}^+, \\ [D_{2m+1}^-, D_{2n}^\pm] &= \mp D_{2(n+m)+1}^-, \\ [D_{2m}^+, D_{2n}^\pm] &= [D_{2m}^-, D_{2n}^\pm] = 0. \end{aligned} \quad (3.34)$$

This set of operators gives a representation of the Lie superalgebra generated by the super vector fields,

$$\begin{cases} V_{2n-1}^+ = \partial_\xi \partial_x^{n-1}, \\ V_{2n-1}^- = \xi \partial_x^n, \end{cases} \quad \begin{cases} V_{2n}^+ = \partial_\xi \xi \partial_x^n, \\ V_{2n}^- = \xi \partial_\xi \partial_x^n, \end{cases} \quad (3.35)$$

$n = 1, 2, \dots$, which are considered as a positive frequency part of the generators of $\widehat{gl1|1}$. It is not hard to see that the series of equations $D_n^\pm S = -(SV_n^\pm S^{-1})_- S$ leads to

$$[D_m^\pm, D_n^{\pm,(\mp)}]_\pm S = (S[V_m^\pm, V_n^{\pm,(\mp)}]_\pm S^{-1})_- S,$$

and hence the integrability requires (3.34), irrespectively of the explicit forms of the operators D_n^\pm .

3.3 Solution of the Grassmann equation

We return to the Grassmann equation (3.3) and try to solve it. The problem is reduced to that of linear algebra by making use of the matrix representation introduced in the previous section. Eq.(3.3) takes the form in the matrix representation [6][‡]

$$\begin{aligned} \sum_{j \leq 1} s_j(x, \xi) \psi[U(t) S_0^{-1}]_{j,m} &= 0, \quad m = 1, 2, \dots \\ \text{with } S(t) &= \sum_{j \leq 1} s_j(x, \xi) D^j, \quad s_0 = 1. \end{aligned} \quad (3.36)$$

From Proposition 1, the matrix $\psi[U(t) S_0^{-1}]$ takes the form

$$\psi[U(t) S_0^{-1}] = e^{x\Lambda + \xi\Gamma} \Phi(t) \Xi_0 e^{-x\Lambda + \xi\Gamma},$$

where we write

$$\Xi_0 = \Xi[S_0^{-1}] \quad \text{and} \quad \Phi(t) = \Xi[U(t)], \quad (3.37)$$

[‡] In writing $U(t)$, we let t express the time dependence symbolically. So hereafter when concerning the maximal SKP hierarchy, t stands for $\{t_n^\pm\}$.

for simplicity, and use this notation in the following. Since $(e^{x\Lambda+\xi\Gamma})_{\geq<} = \mathbf{0}$ and $(e^{x\Lambda+\xi\Gamma})_{<<}$ is invertible, Eq.(3.36) is rewritten as

$$\sum_{j \leq 0} s_j(x, \xi) Z(x, \xi, t)_{j, -m} = 0, \quad m = 1, 2, \dots$$

with $Z(x, \xi, t) = e^{x\Lambda+\xi\Gamma} \Phi(t) \Xi_0.$ (3.38)

To solve Eq.(3.36) or (3.38), let us remind ourselves of some basic ingredients of linear superalgebra. Let $X = (X_{ij})_{i,j \in \mathbf{Z}}$ be an even homogeneous element of $Mat(\mathbf{Z} \times \mathbf{Z}, \mathcal{A})$ i.e. $|X_{ij}| = i + j \pmod{2}$, and let $X_I = (X_{ij})_{i,j \in I \cap \mathbf{Z}}$ be a sub-matrix of X specified by a set of indices $I \subset \mathbf{Z}$. Supermatrices are often written in the block form with respect to the Grassmann parity. Let us associate with X_I a matrix \check{X}_I

$$\check{X}_I = (X_I^{ab})_{a,b=0,1}, \quad X_I^{ab} = (X_{ij})_{i \in \{a+2\mathbf{Z}\} \cap I, j \in \{b+2\mathbf{Z}\} \cap I}. \quad (3.39)$$

When the matrix X_I^{00} (X_I^{11}) is invertible, we define a matrix \widehat{X}_I^{00} (\widehat{X}_I^{11}) by

$$\widehat{X}_I^{00} = X_I^{00} - X_I^{01}(X_I^{11})^{-1}X_I^{10}, \quad \widehat{X}_I^{11} = X_I^{11} - X_I^{10}(X_I^{00})^{-1}X_I^{01}. \quad (3.40)$$

The superdeterminant of the matrix X_I (or \check{X}_I) is defined, as usual, by

$$\text{sdet} X_I = \frac{\det \widehat{X}_I^{00}}{\det X_I^{11}} = \frac{\det X_I^{00}}{\det \widehat{X}_I^{11}}. \quad (3.41)$$

Keeping Eq.(3.36) in mind, let us study an equation with a given matrix $X_{\leq\leq}$

$$v_{\leq} X_{\leq\leq} = \mathbf{0}, \quad (3.42)$$

where $v_{\leq} = (v_j)_{j \leq 0}$ is a semi-infinite row vector with a restriction $v_0 = 1$. If both of the matrices $X_{\leq\leq}$ and $X_{<<}$ are regular (i.e. $\det X_{\leq\leq}^{00} \neq 0$, $\det X_{<<}^{00} \neq 0$ and $\det X_{\leq\leq}^{11} = \det X_{<<}^{11} \neq 0$), a unique solution for v_{\leq} is given by

$$v_j = \frac{\text{sdet} X_{\leq\leq}}{\text{sdet} X_{<<}} (X_{\leq\leq}^{-1})_{0j}, \quad j = 0, -1, -2, \dots \quad (3.43)$$

It is evident that v_{\leq} given above satisfies Eq.(3.42), and $v_0 = 1$ results from

$$(X_{\leq\leq}^{-1})_{00} = (\widehat{X}_{\leq\leq}^{00})^{-1}_{00} = \frac{\det \widehat{X}_{<<}^{00}}{\det \widehat{X}_{\leq\leq}^{00}} = \frac{\text{sdet} X_{<<}}{\text{sdet} X_{\leq\leq}},$$

here the last equality holds because in our notation $X_{\leq\leq}^{11} = X_{<<}^{11}$. Of course the right hand side of (3.43) depends only on the part $X_{<<}$. In (3.43), $\text{sdet} X_{\leq\leq} \cdot (X_{\leq\leq}^{-1})_{0j}$ is considered as a $(0, j)$ -cofactor and is expressed in terms of superdeterminants as shown in [6], affording the supersymmetric generalization of Cramer's formula. However the

expression (3.43) is more convenient for later use. Returning to (3.38) we observe that $Z(x, \xi, t)|_{x=\xi=t=0} = \Xi_0$ and $(\Xi_0)_{ij} = \delta_{ij}$, for $j \geq i$, and thus we can conclude that formally $Z_{\leq\leq}$ and $Z_{<<}$ are both regular. Consequently, Eq.(3.38) has a unique solution

$$s_j = \frac{\text{sdet} Z_{\leq\leq}}{\text{sdet} Z_{<<}} (Z_{\leq\leq}^{-1})_{0j}, \quad j = 0, -1, -2, \dots \quad (3.44)$$

The concrete expressions of $\psi[U(t)]$ for each SKP hierarchies are calculated as

$$\psi[U_{MR}(t)] = \exp \sum_{l=1} \left(t_{2l-1} J \Gamma^{2l-1} + t_{2l} \Lambda^l \right), \quad (3.45)$$

$$\psi[U_J(t)] = \exp \sum_{l=1} \left\{ t_{2l-1} \left(\frac{1+J}{2} - \xi \Gamma \right) \Gamma^{2l-1} + t_{2l} \Lambda^l \right\}, \quad (3.46)$$

$$\begin{aligned} \psi[U_M(t^\pm)] &= \exp \sum_{l=0} \left\{ t_{2l+1}^+ \left(\frac{1+J}{2} - \xi \Gamma \right) \Gamma^{2l+1} + t_{2l+1}^- \left(-\frac{1-J}{2} + \xi \Gamma \right) \Gamma^{2l+1} \right\} \\ &\quad \times \exp \sum_{l=1} \left\{ t_{2l}^+ \left(\frac{1+J}{2} - \xi J \Gamma \right) \Lambda^l + t_{2l}^- \left(\frac{1-J}{2} + \xi J \Gamma \right) \Lambda^l \right\}, \end{aligned} \quad (3.47)$$

Here we note $\Phi(t) \equiv \Xi[U(t)] = \psi[U(t)]|_{\xi=0}$.

3.4 Super-wave function

The notion of the super-wave function (or Baker-Akhiezer function) arises naturally from Eq.(3.3). Taking Proposition 3 (ii) into account Eq.(3.3) is rewritten as

$$< S(t)U(t)e^{xz+\xi\theta}, \rho(S_0^{-1})\zeta^m > = 0, \quad m = 0, 1, 2, \dots \quad (3.48)$$

Here

$$w(x, \xi, t; \zeta) \equiv S(t)U(t)e^{xz+\xi\theta} \quad (3.49)$$

is the super-wave function of the SKP hierarchy. Remember that

$$v^{(m)} = \rho(S_0^{-1})\zeta^m = \sum_j \zeta^j (F_0)_{jm}, \quad m = 0, 1, 2, \dots, \quad \text{with} \quad F_0 = JK\Xi_0 KJ$$

are the basis vectors (2.19) of the subspace $W_{S_0} = \sigma(S_0) \in \Gamma_0$. Therefore in terms of the wave function w of the form (3.49), Eq.(3.3) is equivalently represented as a statement

$$w(x, \xi, t; \zeta) \in W_{S_0}^\perp. \quad (3.50)$$

where $W_{S_0}^\perp$ is the subspace orthogonal to W_{S_0} with respect to the pairing (2.36).

From (3.7) and (3.5), the super-wave function satisfies

$$dw = (S\Omega S^{-1})_+ w. \quad (3.51)$$

Let us denote the factor $U(t) e^{xz+\xi\theta}$ by $A(x, \xi, t; \zeta)$. Consider the maximal SKP hierarchy and define formal power series

$$\begin{aligned} t_e^\pm(z) &= \sum_{m=1} t_{2m}^\pm z^m, \\ t_e(z) &= \frac{t_e^+(z) + t_e^-(z)}{2}, \quad \bar{t}_e(z) = \frac{t_e^+(z) - t_e^-(z)}{2}, \end{aligned} \quad (3.52)$$

$$\begin{aligned} t_o^\pm(z) &= \sum_{m=0} t_{2m+1}^\pm z^m, \\ t_o(z) &= \frac{t_o^+(z) + t_o^-(z)}{2}, \quad \bar{t}_o(z) = \frac{t_o^+(z) - t_o^-(z)}{2}, \end{aligned} \quad (3.53)$$

$$t_\xi^\pm(z) = t_o^\pm(z) \pm \xi. \quad (3.54)$$

Then we have

$$\begin{aligned} A_M(x, \xi, t^\pm; \zeta) &= U_{(1)} U_{(0)} \exp(xz + \xi\theta) \\ &= \exp \left\{ t_o^+(z) \partial_\xi + z t_o^-(z) \xi \right\} \exp \left\{ t_e^+(z) \partial_\xi \xi + t_e^-(z) \xi \partial_\xi \right\} \exp(xz + \xi\theta) \\ &= \exp xz \cdot \left\{ \exp \left\{ t_e^+(z) - \frac{1}{2} z \left(t_o^+ t_o^-(z) + 2\xi t_o^-(z) \right) \right\} \right. \\ &\quad \left. + \exp \left\{ t_e^-(z) + \frac{1}{2} z t_o^+ t_o^-(z) \right\} (t_o^+(z) + \xi) \theta \right\} \\ &= \exp \left\{ xz + t_e^+(z) + \xi z \bar{t}_o(z) + t_\xi^+(z) \left(\theta \exp(-2\bar{t}_e(z)) - \frac{1}{2} z t_\xi^-(z) \right) \right\}, \end{aligned} \quad (3.55)$$

and

$$w_M(x, \xi, t^\pm; \zeta) = S(t^\pm) A_M(x, \xi, t^\pm; \zeta). \quad (3.56)$$

The wave functions for JSKP and MRSKP hierarchies[3] are obtained from (3.55) and (3.56) by replacing the time variables as in (3.16). For JSKP, putting $t_o^+(z) \rightarrow t_o(z)$, $t_o^-(z) = 0$, $\bar{t}_o(z) \rightarrow \frac{1}{2} t_o(z)$ and $t_e^+(z) = t_e^-(z) = t_e(z)$, we have

$$w_J(x, \xi, t; \zeta) = S(t) \exp(xz + \xi\theta + t_e(z) + t_o(z)\theta), \quad (3.57)$$

and for MRSKP, putting $t_o^+(z) = t_o^-(z) = t_o(z)$ and $t_e^+(z) = t_e^-(z) = t_e(z)$, we have

$$w_{MR}(x, \xi, t; \zeta) = S(t) \exp \left\{ xz + \xi(\theta - z t_o(z)) + t_e(z) + t_o(z)\theta \right\}. \quad (3.58)$$

Here we make an additional remark. From (2.39) we notice that, for $P = \sum_j p_j D^j \in \mathcal{E}$,

$$P e^{xz+\xi\theta} = \sum_{j,m} p_j (e^{x\Lambda+\xi\Gamma})_{jm} \zeta^m, \quad (3.59)$$

and hence for $P, Q \in \mathcal{E}$,

$$P Q e^{xz+\xi\theta} = \sum_{j,m} p_j (\psi[Q] e^{x\Lambda+\xi\Gamma})_{jm} \zeta^m = \sum_{j,m} p_j (e^{x\Lambda+\xi\Gamma} \Xi[Q])_{jm} \zeta^m. \quad (3.60)$$

Thus the wave function w has the expression

$$w(x, \xi, t; \zeta) = \sum_m w_m \zeta^m \quad \text{with} \quad w_m = \sum_j s_j (e^{x\Lambda+\xi\Gamma} \Phi(t))_{jm}. \quad (3.61)$$

3.5 Dual wave function and the bilinear identity

Let us introduce an operation \star for invertible SΨDO's by

$$P^\star \equiv (P^{-1})^\star = (P^\star)^{-1}. \quad (3.62)$$

For example, consider this operation on the set of even monic SΨDO's denoted by \mathcal{E}_0^\times . Then \star defines an endomorphism of \mathcal{E}_0^\times . We also define an operation \star for invertible (even homogeneous) matrices $X \in \text{Mat}(\mathbf{Z} \times \mathbf{Z}, \mathcal{A})$ by

$$X^\star \equiv IK(X^{\text{st}})^{-1}KI. \quad (3.63)$$

Then, from (2.42) we have

$$\psi[P^\star] = \psi[P]^\star, \quad P \in \mathcal{E}_0^\times. \quad (3.64)$$

Applying the operation \star to (3.3), we obtain the equation for $S^\star(t)$:

$$(S^\star(t)U^\star(t)S_0^{\star-1})_- = 0. \quad (3.65)$$

As in the case of wave function, we can write this equation in the form

$$\langle S^\star(t)U^\star(t)e^{-(xz+\xi\theta)}, \rho(S_0^{\star-1})\zeta^m \Big|_{\substack{z \rightarrow -z \\ \theta \rightarrow -\theta}} \rangle = 0, \quad m = 0, 1, 2, \dots \quad (3.66)$$

Here, for later convenience we put, in the integral defining the pairing, $z \rightarrow -z$, $\theta \rightarrow -\theta$ (i.e., $\zeta^m \rightarrow (-1)^{\frac{m(m+1)}{2}}\zeta^m = I_{mm}\zeta^m$, $m \in \mathbf{Z}$). The dual wave function is given by

$$\tilde{w}(x, \xi, t; \zeta) = S^\star(t)U^\star(t)e^{-(xz+\xi\theta)}. \quad (3.67)$$

Let us calculate the factor $\tilde{A} \equiv U^\star(t)e^{-(xz+\xi\theta)}$. For the maximal SKP hierarchy, we have

$$\begin{aligned} \tilde{A}_M(x, \xi, t_{2n+1}^+, t_{2n+1}^-, t_{2n}^+, t_{2n}^-; \zeta) &= U_{(1)}^\star(t)U_{(0)}^\star(t) \exp\{-(xz + \xi\theta)\} \\ &= \exp \sum_{n=0} \left\{ t_{2n+1}^+ \partial_\xi (-\partial_x)^n - t_{2n+1}^- \xi (-\partial_x)^{(n+1)} \right\} \\ &\quad \times \exp - \sum_{n=0} \left\{ t_{2n}^+ \xi \partial_\xi (-\partial_x)^n + t_{2n}^- \partial_\xi \xi (-\partial_x)^n \right\} \exp\{-(xz + \xi\theta)\} \\ &= \exp \sum_{n=0} \left\{ -t_{2n+1}^+ \partial_\xi \partial_x^n + t_{2n+1}^- \xi \partial_x^{(n+1)} \right\} \\ &\quad \times \exp - \sum_{n=0} \left\{ t_{2n}^- \partial_\xi \xi \partial_x^n + t_{2n}^+ \xi \partial_\xi \partial_x^n \right\} \exp(xz + \xi\theta) \Big|_{\substack{x \rightarrow -x \\ \xi \rightarrow -\xi}} \\ &= A_M(-x, -\xi, -t_{2n+1}^+, t_{2n+1}^-, -t_{2n}^+, -t_{2n}^-; \zeta) \\ &= \exp \left\{ -xz - t_e^-(z) - \xi z \bar{t}_e(z) - t_\xi^+(z) \left(\theta \exp(-2\bar{t}_e(z)) - \frac{1}{2} z t_\xi^-(z) \right) \right\} \\ &= \exp 2\bar{t}_e(z) \cdot A_M(x, \xi, t^\pm; \zeta)^{-1}. \end{aligned} \quad (3.68)$$

In (3.66) we see

$$\tilde{v}^{(m)} \equiv (-1)^{\frac{m(m+1)}{2}} \rho(S_0^{\star-1}) \zeta^m \Big|_{\substack{z \rightarrow -z \\ \theta \rightarrow -\theta}} = \sum_j \zeta^j (\tilde{F}_0)_{jm} \quad \text{with} \quad \tilde{F}_0 = IJK\Xi_0^{\star}KJI. \quad (3.69)$$

Then from (3.63),

$$\tilde{F}_0 = J(\Xi_0^{\text{st}})^{-1}J. \quad (3.70)$$

\tilde{F}_0 is a triangular matrix $(\tilde{F}_0)_{ij} = \delta_{ij}$, for $j \geq m$, inheriting the triangularity from $\Xi_0 = \Xi[S_0^{-1}]$. Therefore the matrix $(\tilde{F}_0)_{\geq}$ can be considered as a frame matrix of a subspace in \mathcal{H} , which we denote by \tilde{W}_{S_0} , defining an element of the super-Grassmannian Γ_0 . Then Eq.(3.66) asserts

$$\tilde{w}(x, \xi, t; \zeta) \in \tilde{W}_{S_0}^{\perp}. \quad (3.71)$$

The validity of the bilinear identity for the (MR)SKP hierarchy has been established in [6]. Actually its validity does not depend on the explicit forms of the time evolution operators. For completeness we state the bilinear identity with its proof. Consider the SKP hierarchy whose time evolution operator is $U(t)$.

Theorem 2 *Let $w(x, \xi, t; \zeta) = PA(x, \xi, t; \zeta)$ and $\tilde{w}(x, \xi, t; \zeta) = Q\tilde{A}(x, \xi, t; \zeta)$, $P, Q \in \Gamma_0$, where $A = Ue^{xz+\xi\theta}$ and $\tilde{A} = U^{\star}e^{-(xz+\xi\theta)}$. The necessary and sufficient condition for w and \tilde{w} being the wave function and its dual for the SKP hierarchy concerned is that they satisfy the following bilinear identity:*

$$\oint dzd\theta w(x, \xi, t; \zeta) \tilde{w}(x', \xi', t'; \zeta) = 0. \quad (3.72)$$

[Proof] (Necessity). Let $P = Q^{\star} = S$ and set $v^{(m)} = \sum_j \zeta^j (F_0)_{jm}$ and $\tilde{v}^{(m)} = \sum_j \zeta^j (\tilde{F}_0)_{jm}$ as before. In addition to the fact that $\{v^{(m)}\}_{m \in \mathbf{Z}_{\geq}}$ and $\{\tilde{v}^{(m)}\}_{m \in \mathbf{Z}_{\geq}}$ are the bases of W_{S_0} and \tilde{W}_{S_0} respectively, $\{v^{(m)}\}_{m \in \mathbf{Z}}$ and $\{\tilde{v}^{(m)}\}_{m \in \mathbf{Z}}$ are both considered as bases of \mathcal{H} since F_0 and \tilde{F}_0 are both triangular matrices with unit diagonal elements. Then

$$\langle \tilde{v}^{(j)}, v^{(m)} \rangle = (J\tilde{F}_0^{\text{st}}JKF_0)_{jm} = K_{jm} = \delta_{j+m+1,0}, \quad (3.73)$$

which says

$$W_{S_0}^{\perp} = \tilde{W}_{S_0} \quad \text{and} \quad \tilde{W}_{S_0}^{\perp} = W_{S_0}. \quad (3.74)$$

Therefore, from (3.50) and (3.71), we find the bilinear identity.

(Sufficiency). From Proposition 3 (iii), we have $(P(t)U(t)U(t')^{-1}Q(t')^{\star})_- = 0$. Putting $t_n = t'_n$ and noting $P, Q \in \Gamma_0$, we see $Q = P^{\star}$. Moreover from the bilinear identity,

$$\begin{aligned} 0 &= \oint dzd\theta \partial_{t_n} w(x, \xi, t; \zeta) \tilde{w}(x', \xi', t'; \zeta) \\ &= \oint dzd\theta (B_n w(x, \xi, t; \zeta)) \tilde{w}(x', \xi', t'; \zeta), \end{aligned}$$

where $B_n = (\partial_{t_n} P)P^{-1} + P\Omega_n P^{-1}$. Again from Proposition 3 (iii), we have

$$((B_n P U)(t)(U^{-1} Q^*)(t'))_- = 0.$$

Putting $t_n = t'_n$ together with the fact $Q = P^*$, we see $(B_n)_- = 0$, from which follows the Sato equations $\partial_{t_n} P = -(P\Omega_n P^{-1})_- P^{-1}$. \square

Let us make a few remarks on the dual wave function.

1. Let

$$S^*(t) = \sum_{j \leq 0} (-1)^{\frac{j(j+1)}{2}} \tilde{s}_j D^j. \quad (3.75)$$

From (3.67) and taking account of (3.60), we see the dual wave function has an expression

$$\begin{aligned} \tilde{w}(x, \xi, t; \zeta) &= \sum_{j, m} \tilde{s}_j (e^{-(x\Lambda + \xi J\Gamma)} I \Phi(t)^* I)_{jm} \zeta^m \\ &= \sum_{j, m} (-1)^m \zeta^m (e^{-(x\Lambda + \xi J\Gamma)} I \Phi(t)^* I)_{mj}^{\text{st}} \tilde{s}_j \\ &= \sum_{j, m} \zeta^{-(m+1)} (J e^{-(x\Lambda + \xi J\Gamma)} \Phi(t)^{-1} J)_{mj} (J \tilde{s})_{-(j+1)}. \end{aligned} \quad (3.76)$$

To obtain the above expression, we use the identities $(\xi\Gamma)^{\text{st}} = K\xi J\Gamma K$, $\Lambda^{\text{st}} = K\Lambda K$, $I\Gamma I = -J\Gamma$ and $I\Lambda I = -\Lambda$.

2. As the counterpart of Eq.(3.38), rewriting (3.65) as before, we have

$$\sum_{j \leq 0} (IZ^* I)_{mj}^{\text{st}} \tilde{s}_j = 0, \quad m = -1, -2, \dots, \quad \text{with} \quad Z = e^{x\Lambda + \xi\Gamma} \Phi(t) \Xi_0. \quad (3.77)$$

We can solve this equation in terms of the matrix Z as follows. Noting $K^{\text{st}} = JK$ (K is a parity odd matrix), we see from the definition of the \star operation

$$(IZ^* I)^{\text{st}} K Z = -K, \quad (3.78)$$

and hence especially

$$\sum_{j \in \mathbf{Z}} (IZ^* I)_{mj}^{\text{st}} Z_{-(j+1), k} = 0, \quad \text{for } m < 0 \text{ and } k < 0. \quad (3.79)$$

Let us consider the $\mathbf{N} \times \mathbf{N}$ matrix

$$Z_{<<}^{(j)} \equiv \begin{pmatrix} Z_{j<} \\ Z_{<<} \end{pmatrix}, \quad (3.80)$$

where the symbol \ll represents indices running on $\{-2, -3, \dots\}$. In other words, $Z_{<<}^{(j)}$ is a matrix $Z_{<<}$ except its -1 st row is replaced with the j -th row of the matrix $Z_{<} = \begin{pmatrix} Z_{\geq <} \\ Z_{<<} \end{pmatrix}$. Then from (3.79) we have

$$\begin{aligned} 0 &= \sum_{j \in \mathbf{Z}} (IZ^*I)_{mj}^{\text{st}} \cdot \text{sdet}_{\pi} Z_{<<}^{(-j-1)} \\ &= \sum_{j \leq 0} (IZ^*I)_{mj}^{\text{st}} \cdot \text{sdet}_{\pi} Z_{<<}^{(-j-1)}, \end{aligned} \quad (3.81)$$

where sdet_{π} is defined, using the previous notations, as

$$\text{sdet}_{\pi} X_I = \frac{\det \widehat{X}_I^{11}}{\det X_I^{00}} \quad (= (\text{sdet} X_I)^{-1}). \quad (3.82)$$

The second equality in (3.81) follows because of the known property of the (super) determinant:

$$\text{sdet}_{\pi} \begin{pmatrix} Z_{j<} \\ Z_{\ll} \end{pmatrix} = 0, \quad \text{for } j \in \{-2, -3, \dots\}.$$

Now from (3.81) we have a solution of (3.77) normalized by $\tilde{s}_0 = 1$:

$$\tilde{s}_{-j} = \frac{1}{\text{sdet}_{\pi} Z_{<<}} \text{sdet}_{\pi} Z_{<<}^{(j-1)}, \quad j = 0, 1, 2, \dots \quad (3.83)$$

4 The operator theory for the SKP hierarchies

In this section we reformulate the results in the preceding sections in the field theoretic language.

4.1 B-C system

The B-C system[18] consists of a pair of superfields B and C that take the forms

$$B(z, \theta) = \beta(z) + \theta b(z), \quad C(z, \theta) = c(z) + \gamma(z)\theta. \quad (4.1)$$

We set, for the time being, the superconformal weights of B and C to be 0 and $\frac{1}{2}$, respectively. The mode expansions of the fields are

$$\begin{aligned} B(z, \theta) &= \sum_{j \in \mathbf{Z}} (\beta_{j+\frac{1}{2}} + \theta b_j) z^{-j-1} = \sum_{j \in \mathbf{Z}} \zeta^{-j-1} B_j, \\ C(z, \theta) &= \sum_{j \in \mathbf{Z}} (c_j + \gamma_{j-\frac{1}{2}} \theta) z^{-j} = \sum_{j \in \mathbf{Z}} C_j \zeta^{-j}. \end{aligned} \quad (4.2)$$

Here we set

$$\begin{cases} B_{2j} &= b_j, \\ B_{2j+1} &= \beta_{j+\frac{1}{2}}, \end{cases} \quad \text{and} \quad \begin{cases} C_{2j} &= c_j, \\ C_{2j+1} &= \gamma_{j+\frac{1}{2}}. \end{cases} \quad (4.3)$$

The oscillators satisfy the (anti-)commutation relations

$$\begin{aligned} [C_m, B_n]_{\pm} &= C_m B_n - (-1)^{mn} B_n C_m = \delta_{m+n,0}, \\ [C_m, C_n]_{\pm} &= [B_m, B_n]_{\pm} = 0, \end{aligned} \quad (4.4)$$

and generate a non-commutative algebra (super Weyl algebra) denoted by \mathbf{A}_{BC} . \mathbf{A}_{BC} is a tensor product of the Clifford algebra \mathbf{A}_{bc} generated by the oscillators b_n and c_n and the Weyl algebra $\mathbf{A}_{\beta\gamma}$ generated by the oscillators $\beta_{n+\frac{1}{2}}$ and $\gamma_{n+\frac{1}{2}}$:

$$\mathbf{A}_{BC} = \mathbf{A}_{bc} \otimes \mathbf{A}_{\beta\gamma}. \quad (4.5)$$

Let us define a set of current operators

$$\begin{aligned} j^+(z) &= \lim_{w \rightarrow z} \left\{ -b(z)c(w) + \frac{1}{z-w} \right\} = \sum_{n \in \mathbf{Z}} j_n^+ z^{-n-1}, \\ j^-(z) &= \lim_{w \rightarrow z} \left\{ -\beta(z)\gamma(w) - \frac{1}{z-w} \right\} = \sum_{n \in \mathbf{Z}} j_n^- z^{-n-1}, \\ \psi^+(z) &= c(z)\beta(z) = \sum_{n \in \mathbf{Z}} \psi_{n+\frac{1}{2}}^+ z^{-n-1}, \\ \psi^-(z) &= \gamma(z)b(z) = \sum_{n \in \mathbf{Z}} \psi_{n+\frac{1}{2}}^- z^{-n-2}. \end{aligned} \quad (4.6)$$

These currents form generators of a super Lie algebra $\widehat{gl|1|1}$. We denote its envelope (i.e. super Heisenberg algebra) by $\mathbf{A}_{j\psi}$:

$$\begin{aligned} [j_m^{\pm}, j_n^{\pm}] &= \pm m \delta_{m+n,0}, \\ [j_m^+, j_n^-] &= 0, \end{aligned} \quad (4.7)$$

$$\begin{aligned} [j_m^{\pm}, \psi_r^+] &= \pm \psi_{m+r}^+, \\ [j_m^{\pm}, \psi_r^-] &= \mp \psi_{m+r}^-, \end{aligned} \quad (4.8)$$

$$\begin{aligned} \{\psi_r^+, \psi_s^-\} &= \left(-r + \frac{1}{2}\right) \delta_{r+s,0} - j_{r+s}^t, \quad \text{with} \quad j_m^t = j_m^+ + j_m^-, \\ \{\psi_r^{\pm}, \psi_s^{\pm}\} &= 0. \end{aligned} \quad (4.9)$$

Note that the total current of the system $j^t(z) \equiv j^+(z) + j^-(z)$ satisfies

$$[j^t(z), j^t(w)] = 0 \quad \text{and} \quad [j^t(z), \psi^{\pm}(w)] = 0, \quad (4.10)$$

and especially j_0^t is a central element of $\mathbf{A}_{j\psi}$.

Next, we recall the Fock representations of the B-C system. Let us define the vacuum state $|0\rangle = |0\rangle_{bc} \otimes |0\rangle_{\beta\gamma}$ by

$$C_m |0\rangle = 0, \quad \text{for } m > 0, \quad \text{and} \quad B_m |0\rangle = 0, \quad \text{for } m \geq 0 \quad (4.11)$$

and consider the Fock space (i.e. left \mathbf{A}_{BC} -module) generated from this state. We denote it by $\mathcal{F}_{BC}^{(0)}$. One can also consider the charged vacuum states

$$|(p, q)\rangle = |p\rangle_{bc} \otimes |q\rangle_{\beta\gamma}, \quad p, q \in \mathbf{Z}, \quad (4.12)$$

where the states $|p\rangle_{bc}$ and $|q\rangle_{\beta\gamma}$ are specified by

$$\left. \begin{array}{l} \text{for } n > -p, \quad c_n \\ \text{for } n \geq p, \quad b_n \end{array} \right\} |p\rangle_{bc} = 0 \quad \text{and} \quad \left. \begin{array}{l} \text{for } r > p, \quad \gamma_r \\ \text{for } r > -p, \quad \beta_r \end{array} \right\} |p\rangle_{\beta\gamma} = 0. \quad (4.13)$$

These state have expressions such as

$$|p\rangle_{bc} = \begin{cases} c_{-(p-1)} \cdots c_{-1} c_0 |0\rangle_{bc} & \text{for } p > 0, \\ b_p \cdots b_{-2} b_{-1} |0\rangle_{bc} & \text{for } p < 0 \end{cases} \quad (4.14)$$

$$|p\rangle_{\beta\gamma} = \begin{cases} \delta(\beta_{-p+\frac{1}{2}}) \cdots \delta(\beta_{-\frac{3}{2}}) \delta(\beta_{-\frac{1}{2}}) |0\rangle_{\beta\gamma} & \text{for } p > 0, \\ \delta(\gamma_{p+\frac{1}{2}}) \cdots \delta(\gamma_{-\frac{3}{2}}) \delta(\gamma_{-\frac{1}{2}}) |0\rangle_{\beta\gamma} & \text{for } p < 0. \end{cases} \quad (4.15)$$

The regularization of the current j^\pm appeared in (4.6) are defined so that $j_0^+ |0\rangle_{bc} = 0$ and $j_0^- |0\rangle_{\beta\gamma} = 0$, and then the state $|(p, q)\rangle$ possesses the following properties for the current algebra $\mathbf{A}_{j\psi}$:

$$\begin{aligned} (j_n^+ - p\delta_{n0})|(p, q)\rangle &= 0, & \text{for } n \geq 0, \\ (j_n^- - q\delta_{n0})|(p, q)\rangle &= 0, & \text{for } n \geq 0, \\ \psi_r^+ |(p, q)\rangle &= 0, & \text{for } r > -(p+q), \\ \psi_r^- |(p, q)\rangle &= 0, & \text{for } r > p+q-1. \end{aligned} \quad (4.16)$$

Let us consider the Fock representation of \mathbf{A}_{BC} generated from $|(p, q)\rangle$. Since

$$\mathbf{A}_{bc}|p\rangle_{bc} = \mathbf{A}_{bc}|0\rangle_{bc},$$

we have a unique b-c Fock representation $\mathcal{F}_{bc} \equiv \mathbf{A}_{bc}|0\rangle_{bc}$. On the other hand, since

$$\mathbf{A}_{\beta\gamma}|p\rangle_{\beta\gamma} \neq \mathbf{A}_{\beta\gamma}|q\rangle_{\beta\gamma} \quad \text{for } p \neq q,$$

we have inequivalent Fock representations $\mathcal{F}_{\beta\gamma}^{(p)} = \mathbf{A}_{\beta\gamma}|p\rangle_{\beta\gamma}$, for every integer p called “picture charge” [18]. Therefore the Fock representations of the total B-C system are given by

$$\mathcal{F}_{BC}^{(p)} \equiv \mathcal{F}_{bc} \otimes \mathcal{F}_{\beta\gamma}^{(p)}, \quad p \in \mathbf{Z}. \quad (4.17)$$

Actually, to realize the super-Grassmannian within the language of the B-C system, it suffices for us to work on the 0-th picture $\mathcal{F}_{BC}^{(0)}$.

To obtain correlation functions, we need to prepare bra-states. The vacuum bra-states are defined through the involution \dagger in \mathbf{A}_{BC} :

$$\begin{aligned} b_n^\dagger &= b_{-n}, & \gamma_r^\dagger &= \gamma_{-r}, \\ c_n^\dagger &= c_{-n}, & \beta_r^\dagger &= -\beta_{-r}. \end{aligned} \quad (4.18)$$

Then we see that

$$(j_n^\pm)^\dagger = -j_{-n}^\pm - Q^\pm \delta_{n,0}, \quad \psi_r^{\pm\dagger} = \psi_{-r}^\pm, \quad (4.19)$$

where Q^\pm are the so called background charges which in our case are given by

$$Q^+ = -1 \quad \text{and} \quad Q^- = 0. \quad (4.20)$$

We set

$$\langle(p, q)| = (|p, q\rangle)^\dagger. \quad (4.21)$$

Then we have

$${}_{bc}\langle p| j_0^+ = -(p-1) {}_{bc}\langle p| \quad \text{and} \quad {}_{\beta\gamma}\langle q| j_0^- = -q {}_{\beta\gamma}\langle q|. \quad (4.22)$$

The relative normalizations of the bra- and ket-vacuum states are fixed by

$${}_{bc}\langle -p+1|p\rangle_{bc} = 1 \quad \text{and} \quad {}_{\beta\gamma}\langle -q|q\rangle_{\beta\gamma} = 1. \quad (4.23)$$

4.2 Representation of solutions for the Grassmann equation

We know that the Fock space of the theory of free fermion such as the b-c system is constructed in terms of semi-infinite wedge products (or semi-infinite products of Grassmann odd variables). We can extend this view in a supersymmetric way. Let $\tilde{C}_{2n} = \tilde{c}_n$ and $\tilde{C}_{2n+1} = \tilde{\gamma}_{n+\frac{1}{2}}$, $n \in \mathbf{Z}$ be infinitely many odd and even variables, respectively. The algebra \mathbf{A}_{BC} is represented through

$$C_j \Rightarrow \tilde{C}_j \quad \text{and} \quad B_j \Rightarrow (-1)^j \partial_{\tilde{C}_{-j}}, \quad (4.24)$$

and then the vacuum states are expressed as semi-infinite products of delta functions:

$$|p, q\rangle = \prod_{j=1-p}^{\infty} \tilde{c}_j \prod_{k=q}^{\infty} \delta(\tilde{\gamma}_{k+\frac{1}{2}}). \quad (4.25)$$

Hereafter we use the notation $\prod_{j=p}^{\infty} \tilde{c}_j$ on the understandings that the indices of the odd variables are increasing from left to right in its products:

$$\prod_{j=p}^{\infty} \tilde{c}_j = \tilde{c}_p \tilde{c}_{p+1} \tilde{c}_{p+2} \cdots$$

Note that for odd variables $\tilde{c}_n = \delta(\tilde{c}_n)$.

One can observe easily that the correlation functions are calculated formally by multiple integrations:

$$\begin{aligned} & \langle(p', q')| f(C_i, B_j) |p, q\rangle \\ &= \int \prod_{r>-q'} d\tilde{\gamma}_r \int \prod_{l\geq p'} d\tilde{c}_l f(\tilde{C}_i, (-1)^j \partial_{\tilde{C}_{-j}}) \prod_{j\geq 1-p} \tilde{c}_j \prod_{s>q} \delta(\tilde{\gamma}_s) \Bigg|_{\substack{\tilde{\gamma}_r=0, \quad r<-q', \\ \tilde{c}_l=0, \quad l<p'}.} \end{aligned} \quad (4.26)$$

where $f(C_i, B_j)$ is some polynomial function under consideration and the integration over the odd variables $\int \prod_{j \geq p} d\tilde{c}_j$ is Berezin integral defined as

$$\int \prod_{j \geq p} d\tilde{c}_j h(\tilde{c}_j) = \left(\cdots \partial_{\tilde{c}_{p+2}} \partial_{\tilde{c}_{p+1}} \partial_{\tilde{c}_p} h(\tilde{c}_j) \right). \quad (4.27)$$

We notice that the expression (4.26) is well defined only when $q' = -q$.

Now we link the B-C system with the SKP hierarchy. Our argument here is straightforward. Let us return to Eq.(3.38) and its solution (3.44). Consider a linear transformation

$$\tilde{C}_j \xrightarrow{X} \tilde{C}(X)_j = \sum_k \tilde{C}_k X_{-k, -j}, \quad j \in \mathbf{Z}. \quad (4.28)$$

We associate the matrix $X_{<}$ with a state

$$|X_{<}\rangle = \prod_{j \geq 1} \tilde{c}(X)_j \prod_{s > 0} \delta(\tilde{\gamma}(X)_s). \quad (4.29)$$

Then, using the formula (4.26), we can show that

$$\langle (1, 0) | X_{<} \rangle = \langle (0, 0) | C_0 | X_{<} \rangle = \text{sdet} X_{<<}, \quad (4.30)$$

and moreover

$$\langle (0, 0) | C_j | X_{<} \rangle = \begin{cases} \text{sdet} X_{\leq \leq} \cdot (X_{\leq \leq}^{-1})_{0, -j} & \text{for } j \geq 0, \\ 0 & \text{for } j < 0. \end{cases} \quad (4.31)$$

Thus the solution (3.44) can be expressed as

$$s_{-j}(x, \xi, t) = \frac{1}{\langle (1, 0) | Z_{<}(x, \xi, t) \rangle} \langle (0, 0) | C_j | Z_{<}(x, \xi, t) \rangle, \quad j = 0, 1, 2, \dots \quad (4.32)$$

One way to compute $\langle (0, 0) | C_j | X_{<} \rangle$ is to change the integration variables:

$$\tilde{C}_j \longrightarrow \tilde{C}'_j = \sum_{k \geq 0} \tilde{C}_k X_{-k, -j} + \sum_{k < 0} \tilde{C}_k X_{-k, -j} = \tilde{C}(X)_j, \quad j = 0, 1, 2, \dots, \quad (4.33)$$

where $\tilde{C}_{<}$ are considered as auxiliary constants. Then the Jacobian is given by

$$\int \prod_{r > 0} d\tilde{\gamma}_r \int \prod_{j \geq 0} d\tilde{c}_j = \text{sdet} X_{\leq \leq} \int \prod_{r > 0} d\tilde{\gamma}'_r \int \prod_{j \geq 0} d\tilde{c}'_j. \quad (4.34)$$

Expressing \tilde{C}_j , $j \geq 0$ in the integrand in terms of \tilde{C}'_{\geq} and $\tilde{C}_{<}$, we have for $j \geq 0$,

$$\langle (0, 0) | C_j | X_{<} \rangle = \int \prod_{r > 0} d\tilde{\gamma}_r \int \prod_{k \geq 0} d\tilde{c}_k \tilde{C}_j \prod_{k \geq 1} \tilde{c}(X)_k \prod_{r > 0} \delta(\tilde{\gamma}(X)_r) \Big|_{\tilde{C}_{<} = 0}$$

$$\begin{aligned}
&= \text{sdet} X_{\leq\leq} \int \prod_{r>0} d\tilde{\gamma}'_r \int \prod_{k\geq 1} d\tilde{c}'_k \int d\tilde{c}'_0 \\
&\quad \sum_{k\geq 0} (\tilde{C}'_k - \sum_{l>0} \tilde{C}_{-l} X_{l,-k}) (X_{\leq\leq}^{-1})_{-k,-j} \prod_{k\geq 1} \tilde{c}'_k \prod_{r>0} \delta(\tilde{\gamma}'_r) \Big|_{\tilde{C}_{<}=0} \\
&= \text{sdet} X_{\leq\leq} \cdot (X_{\leq\leq}^{-1})_{0,-j}.
\end{aligned} \tag{4.35}$$

Here we take the convention as if the factor $\int \prod_{j\geq 1} d\tilde{c}_j$ is Grassmann even:

$$\int \prod_{j\geq 1} d\tilde{c}_j \lambda = \lambda \int \prod_{j\geq 1} d\tilde{c}_j \quad \text{for an arbitrary odd const. } \lambda, \tag{4.36}$$

which is equivalent to the requirement

$$\langle (1,0) | \lambda = \lambda \langle (1,0) | \quad \text{and} \quad \lambda | (0,0) \rangle = | (0,0) \rangle \lambda. \tag{4.37}$$

We can also show that the expression (3.83) for the solution of Eq.(3.77) can be reproduced by

$$(-1)^j \tilde{s}_{-j}(x, \xi, t) = \frac{1}{\langle (1,0) | Z_{<}(x, \xi, t) \rangle} \langle (1,0) | \gamma_{\frac{1}{2}} B_{j-1} | Z_{<}(x, \xi, t) \rangle, \quad j = 0, 1, 2, \dots \tag{4.38}$$

The formula above can also be derived from a similar calculus as in (4.35), which seems of some interest, so we present it in Appendix A.

4.3 Lie algebra $a_{\infty|\infty}(\mathcal{A})$

Let us define the superlinear space

$$\mathbf{H} = \left(\bigoplus_{j \in \mathbf{Z}} \mathbf{C} C_j \right) \otimes \mathcal{A}. \tag{4.39}$$

Of course there is correspondence between \mathbf{H} and \mathcal{H} with $C_j \leftrightarrow \zeta^{j-1}$. According to (4.28), consider the $GL(\mathbf{H})$ transformation

$$C(X)_j = \sum_{k \in \mathbf{Z}} C_k X_{-k,-j}. \tag{4.40}$$

So far we do not be careful with the infinite dimensionality. Here we set X belongs $GL(\mathbf{H})$, if and only if

$$X_{ij} = 0, \quad \text{for } |i - j| \gg 1,$$

and X is invertible. The commutation relation (4.4) is then preserved provided the transformation of the oscillators B_j

$$B(X)_j = (JX^{-1}J)_{jk} B_k \tag{4.41}$$

is accompanied. Because of this symmetry we can construct, in the Fock space $\mathcal{F}_{BC}^{(0)}$, representations of $GL(\mathbf{H})$ and the associated Lie algebra $a_{\infty|\infty}(\mathcal{A})$ defined below. First we introduce the Lie superalgebra (over \mathbf{C}) $\bar{a}_{\infty|\infty} = \bar{a}_{\infty|\infty_0} \oplus \bar{a}_{\infty|\infty_1}$:

$$\begin{aligned}\bar{a}_{\infty|\infty} &= \{ X = (X_{ij}) \in Mat(\mathbf{Z} \times \mathbf{Z}, \mathbf{C}) \mid X_{ij} = 0, \text{ for } |i - j| \gg 1 \}, \\ \bar{a}_{\infty|\infty_0} &= \{ X \in \bar{a}_{\infty|\infty} \mid X^{01} = X^{10} = \mathbf{0} \}, \\ \bar{a}_{\infty|\infty_1} &= \{ X \in \bar{a}_{\infty|\infty} \mid X^{00} = X^{11} = \mathbf{0} \}.\end{aligned}\tag{4.42}$$

Then the Lie algebra $\bar{a}_{\infty|\infty}(\mathcal{A})$ is defined as the even part of the tensor product $\bar{a}_{\infty|\infty} \otimes \mathcal{A}$ [16]. Now we associate the quadratic operators with $X \in \bar{a}_{\infty|\infty} \otimes \mathcal{A}$:

$$\begin{aligned}J[X] &= \sum_{j,k} : C_j X_{-j,k} B_k : (-1)^k = \sum_{j \leq k} C_j X_{-j,k} B_k (-1)^k + \sum_{j > k} B_k X_{-j,k} C_j (-1)^{j(j+k)} \\ &= \sum_{j,k} \left\{ C_j X_{-j,k} B_k (-1)^k - \langle (1, 0) \mid C_j X_{-j,k} B_k \mid (0, 0) \rangle (-1)^k \right\}.\end{aligned}\tag{4.43}$$

Then we see $J[X]$ generates the infinitesimal transformation of (4.40) and (4.41):

$$[J[X], C_j] = \sum_k C_k X_{-k, -j}, \quad [J[X], B_j] = - \sum_k (JXJ)_{jk} B_k.\tag{4.44}$$

Commutation relations among the quadratic operators are

$$[J[X], J[Y]] = J[[X, Y]] + C(X, Y),\tag{4.45}$$

where

$$C(X, Y) = \sum_{i \leq -1} \sum_{j \geq 0} \{ X_{ij} Y_{ji} (-1)^i - X_{ji} Y_{ij} (-1)^j \}.\tag{4.46}$$

Note that the right-hand side of (4.46) is a finite sum when $X, Y \in \bar{a}_{\infty|\infty}(\mathcal{A})$. The above commutation relation defines a central extension of $\bar{a}_{\infty|\infty}(\mathcal{A})$:

$$a_{\infty|\infty}(\mathcal{A}) = \bar{a}_{\infty|\infty}(\mathcal{A}) \oplus \mathcal{A}_0 1.\tag{4.47}$$

4.4 Formula for the (dual) super-wave function

The oscillators (with positive frequencies) in the current operators (4.6) are expressed as

$$\begin{aligned}j_n^+ &= \sum_k C_{2(k+n)} B_{-2k} = J \left[\frac{1+J}{2} \Lambda^n \right], \\ j_n^- &= \sum_k -C_{2(k+n)+1} B_{-2k-1} = J \left[\frac{1-J}{2} \Lambda^n \right], \\ \lambda \psi_{n+\frac{1}{2}}^+ &= \sum_k \lambda C_{2(k+n)} B_{-2k+1} = J \left[\lambda \frac{1+J}{2} \Gamma^{2n+1} \right], \\ \lambda \psi_{n+\frac{1}{2}}^- &= \sum_k \lambda C_{2(k+n)+1} B_{-2k} = J \left[\lambda \frac{1-J}{2} \Gamma^{2n+1} \right],\end{aligned}\tag{4.48}$$

all for $n \in \mathbf{Z}_{>}$, where λ is an odd parameter. Therefore, remembering the time evolution matrix $\Phi_M(t^\pm)$ obtained from (3.47) and using the formulas obtained in the previous section, we have

$$\begin{aligned}
\sum_{j,m} C_{-j} (e^{x\Lambda + \xi\Gamma} \Phi_M(t^\pm))_{jm} \zeta^m &= \sum_j C_{-j} D^j A_M(x, \xi, t^\pm; \zeta) \\
&= e^{xj_1^t + \xi\sigma} \hat{\Phi}_M(t^\pm) C(\zeta) \hat{\Phi}_M(t^\pm)^{-1} e^{-(xj_1^t + \xi\sigma)}, \\
\sum_{j,m} \zeta^{-(m+1)} (J e^{-(x\Lambda + \xi\Gamma)} \Phi_M(t^\pm)^{-1} J)_{mj} B_j &= \sum_j (-1)^{\frac{j(j+1)}{2}} B_{j-1} D^{-j} \tilde{A}_M(x, \xi, t^\pm; \zeta) \\
&= e^{xj_1^t + \xi\sigma} \hat{\Phi}_M(t^\pm) B(\zeta) \hat{\Phi}_M(t^\pm)^{-1} e^{-(xj_1^t + \xi\sigma)}, \quad (4.49)
\end{aligned}$$

where

$$\sigma = \psi_{\frac{1}{2}}^+ + \psi_{\frac{1}{2}}^-, \quad j_1^t = j_1^+ + j_1^-, \quad (4.50)$$

and

$$\hat{\Phi}_M(t^\pm) = \exp(\psi^+[t_o^+] - \psi^-[t_o^-]) \exp(j^+[t_e^+] + j^-[t_e^-]). \quad (4.51)$$

Here we use the following notations:

$$j^\pm[t_e^\pm] \equiv \sum_{n=1} t_{2n}^\pm j_n^\pm = \frac{1}{2\pi i} \oint dz t_e^\pm(z) j^\pm(z), \quad (4.52)$$

$$\psi^\pm[t_o^\pm] \equiv \sum_{n=0} t_{2n+1}^\pm \psi_{n+\frac{1}{2}}^\pm = \begin{cases} \frac{1}{2\pi i} \oint dz t_o^+(z) \psi^+(z) & \text{for } \psi^+[t_o^+], \\ \frac{1}{2\pi i} \oint dz z t_o^-(z) \psi^-(z) & \text{for } \psi^-[t_o^-]. \end{cases} \quad (4.53)$$

At this stage we are led to field theoretic expressions for the wave function and the dual wave function from (3.61), (3.76), (4.32) and (4.38), which are the main results of this section:

$$w(x, \xi, t; \zeta) = \frac{1}{\tau(x, \xi, t; \zeta)} \langle (0, 0) | e^{xj_1^t + \xi\sigma} \hat{\Phi}(t) C(\zeta) | \Xi_{0<} \rangle, \quad (4.54)$$

$$\tilde{w}(x, \xi, t; \zeta) = \frac{1}{\tau(x, \xi, t; \zeta)} \langle (1, 0) | \gamma_{\frac{1}{2}} e^{xj_1^t + \xi\sigma} \hat{\Phi}(t) B(\zeta) | \Xi_{0<} \rangle, \quad (4.55)$$

where

$$\tau(x, \xi, t; \zeta) = \langle (1, 0) | Z_<(x, \xi, t) \rangle = \langle (1, 0) | e^{xj_1^t + \xi\sigma} \hat{\Phi}(t) | \Xi_{0<} \rangle \quad (4.56)$$

is considered as the tau function of the SKP hierarchy. However, one needs $\tau_M(t^\pm)$ of the maximal case in order to draw the wave function from the τ -function.

For a given $X \in GL(\mathbf{H})$, we can find an element $\hat{O}(X)$ in \mathbf{A}_{BC} such that

$$\hat{O}(X) C_j \hat{O}(X)^{-1} = C(X)_j \quad \text{and} \quad \hat{O}(X) B_j \hat{O}(X)^{-1} = B(X)_j. \quad (4.57)$$

The associated state $|X_< \rangle$ lies on $GL(\mathbf{H})$ orbit of the vacuum state, i.e., $|X_< \rangle = \hat{O}(X) |(0, 0) \rangle$, and is viewed as a section of the dual determinant bundle DET^* over the Gr_0 [16].

Although we take here a direct way to show that the wave function and its dual can be expressed in such forms of (4.54) and (4.55), this can be simply derived from (4.49) and the fact that (4.54) and (4.55) satisfy the bilinear identity (3.72). Since $\oint dzd\theta C(\zeta) \otimes B(\zeta)$ is $GL(\mathbf{H})$ invariant, we have the super-Plücker equation[17]

$$\oint dzd\theta C(\zeta) |\Xi_{0<} \rangle \otimes B(\zeta) |\Xi_{0<} \rangle = 0, \quad (4.58)$$

from which (3.72) immediately follows.

Let us mention about the characteristics of the known SKP hierarchies which we can see immediately. For JSKP hierarchy, the operator (4.51) reduces

$$\widehat{\Phi}_J(t) = \exp(\psi^+[t_o] + j^t[t_e]), \quad (4.59)$$

and we see

$$\psi^+[t_o] + j^t[t_e] = \frac{1}{2\pi i} \oint dzd\theta \{ t_o(z) + t_e(z)\theta \} J(z, \theta), \quad (4.60)$$

where $J(\zeta)$ is the superconformal current

$$J(\zeta) = - : B(\zeta)C(\zeta) : . \quad (4.61)$$

Therefore the super-wave function (4.54) in this case respects the superconformal structure (i.e. can be viewed as a superconformal field), if the state $|\Xi_{0<} \rangle$ is associated with a super-Riemann surface by the super-Krichever map[22]. On the other hand for MRSKP hierarchy, we have

$$\widehat{\Phi}_{MR}(t) = \exp\{ (\psi^+[t_o] + j^t[t_e]) - \psi^-[t_o] \}. \quad (4.62)$$

Here $\psi^-(z) = b(z)\gamma(z)$ is one of the charged fermionic generators of the $N = 2$ super Virasoro algebra of the B-C system. Therefore deformation of the geometric data through $\psi^-[t_o]$ generates variation in the moduli of the supercurve.[3] Thus the superconformal structure will be violated even if it exists in the initial data, in the context of the $N = 1$ superconformal symmetry.

5 Tau function description of the SKP hierarchy

Our main goal in this section is to express the (dual) wave function from the tau function through vertex operators. To do this we recall, as an intermediate step, the superbosonization scheme introduced by Kac and van de Leur[17]. Then we construct the vertex operators which act to the τ -function and give the effect of inserting B and C fields in the correlation that represents tau function.

5.1 Kac-van de Leur superbosonization

In [17] Kac-van de Leur established a super boson-fermion correspondence between the Clifford (or Weyl) superalgebra and the super Heisenberg algebra $\mathbf{A}_{j\psi}^{\S}$. We describe here their construction heuristically, with notational change pushing forwards that their Weyl superalgebra is just the B-C system. Since the fermionic b, c and bosonic β and γ fields carry their own charges, we have to introduce, in addition to the $\widehat{gl(1|1)}$ currents, the operators $e^{\phi_0^\pm}$, with ϕ_0^\pm being the conjugate operators with j_0^\pm , which produce the charge sector in the Fock spaces:

$$\begin{aligned} [j_0^\pm, e^{\phi_0^\pm}] &= e^{\phi_0^\pm}, \\ [j_0^\pm, e^{\phi_0^\mp}] &= 0 \quad \text{and} \quad [j_n^\pm, e^{\phi_0^\pm}] = [j_n^\pm, e^{\phi_0^\mp}] = 0 \quad \text{for } n \neq 0. \end{aligned} \quad (5.1)$$

In addition $e^{\phi_0^\pm}$ satisfy

$$e^{-\phi_0^+} \psi_r^\pm e^{\phi_0^+} = \psi_{r\pm 1}^\pm, \quad e^{-\phi_0^-} \psi_r^\pm e^{\phi_0^-} = \psi_{r\pm 1}^\pm. \quad (5.2)$$

These relations are most easily recognized from the well known Friedan-Marinec-Shenker (FMS) bosonization formula of the B-C system[18]. Let us introduce the fields

$$\phi_>^\pm(z) = \pm \sum_{n>0} \frac{-1}{n} j_n^\pm z^{-n} \quad \text{and} \quad \phi_<^\pm(z) = \pm \sum_{n>0} \frac{1}{n} j_{-n}^\pm z^n. \quad (5.3)$$

As for the fermionic b, c fields, the superbosonization rule is nothing but the ordinary bosonization:

$$c(z) = e^{\phi_0^+} z^{j_0^+} e^{\phi_<^+(z)} e^{\phi_>^+(z)}, \quad b(z) = e^{-\phi_0^+} z^{-j_0^+} e^{-\phi_<^+(z)} e^{-\phi_>^+(z)}. \quad (5.4)$$

A simple way to see the rules for the bosonic β and γ fields is to make use of the following OPE's:

$$\psi^+(z) b(w) \sim \frac{1}{z-w} \beta(z), \quad \psi^-(z) c(w) \sim \frac{1}{z-w} \gamma(z). \quad (5.5)$$

Putting the expression (5.4) and taking into account the identities

$$e^{\pm\phi_<^+(w)} \psi^\pm(z) e^{\mp\phi_<^+(w)} = \frac{1}{1 - \frac{w}{z}} \psi^\pm(z), \quad \text{for } |z| > |w|, \quad (5.6)$$

$$e^{\pm\phi_0^+} \psi^\pm(z) e^{\mp\phi_0^+} = \frac{1}{z} \psi^\pm(z), \quad (5.7)$$

one finds the following superbosonization rule.

$$\begin{aligned} \gamma(z) &= e^{\phi_0^+} z^{1+j_0^+} e^{\phi_<^+(z)} \psi^-(z) e^{\phi_>^+(z)}, \\ \beta(z) &= e^{-\phi_0^+} z^{1-j_0^+} e^{-\phi_<^+(z)} \psi^+(z) e^{-\phi_>^+(z)}. \end{aligned} \quad (5.8)$$

^{\S}Actually, there is no essential difference between the Clifford and Weyl case.

For our purpose it is enough to know the formulas (5.4) and (5.8). We notice that the current $j^- = - : \beta \gamma : (or \phi_{\leq}^-)$ does not appear explicitly in these formulas. In particular we do not need the operator $e^{\pm \phi_0^-}$ (which changes the picture number), because the charges of the β, γ are carried through ψ^\pm as well. In addition to β and γ , there exist fields which intermediate different Fock spaces (i.e. different pictures) $\mathcal{F}_{\beta\gamma}^{(p)}$ as we have seen in the previous section. So much for the superbosonization. We defer the rest to Appendix B. Next, let us consider the representation of the super-Heisenberg algebra $\mathbf{A}_{j\psi}$ on the Fock space $\mathcal{F}_{BC}^{(0)}$. In [17] it was shown that $\mathcal{F}_{BC}^{(0)}$ is decomposed into a direct sum of irreducible representations of $\mathbf{A}_{j\psi}$ according to the total charge number $m \in \mathbf{Z}$:

$$\mathcal{F}_{BC}^{(0)} = \bigoplus_{m \in \mathbf{Z}} \mathcal{F}_m^{(0)}, \quad \text{with} \quad j_0^t \mathcal{F}_m^{(0)} = m \mathcal{F}_m^{(0)}. \quad (5.9)$$

From (4.16) we observe that the highest state $|m\rangle$ which specifies the representation $\mathcal{F}_m^{(0)}$ is not $|(m, 0)\rangle$ itself but is given by

$$|m\rangle = \begin{cases} \psi_{\frac{1}{2}}^- \cdots \psi_{m-\frac{3}{2}}^- |(m, 0)\rangle = (\gamma_{-\frac{1}{2}})^{m-1} |(1, 0)\rangle, & \text{for } m \geq 1, \\ \psi_{\frac{1}{2}}^+ \cdots \psi_{|m|-\frac{1}{2}}^+ |(m, 0)\rangle = (\beta_{-\frac{1}{2}})^{|m|} |(0, 0)\rangle, & \text{for } m \leq 0. \end{cases} \quad (5.10)$$

Then the state $|m\rangle$, $m \in \mathbf{Z}$ satisfies the highest weight conditions

$$\begin{aligned} j_0^+ |m\rangle &= \begin{cases} |m\rangle, & \text{for } m \geq 1, \\ 0, & \text{for } m \leq 0, \end{cases} & j_0^- |m\rangle &= \begin{cases} (m-1)|m\rangle, & \text{for } m \geq 1, \\ m|m\rangle, & \text{for } m \leq 0, \end{cases} \\ j_n^\pm |m\rangle &= 0, \quad \text{for } n > 0 \quad \text{and} \quad \psi_r^\pm |m\rangle = 0, \quad \text{for } r > 0. \end{aligned} \quad (5.11)$$

The highest weight states for the right $\mathbf{A}_{j\psi}$ -modules are similarly defined through $\langle m| = (|m\rangle)^\dagger$. We notice that the states $\langle 1| (|1\rangle)$ and $\langle 0| (|0\rangle)$ satisfy the additional conditions respectively,

$$\begin{aligned} \langle 1| \psi_{\frac{1}{2}}^+ &= 0 \quad (\psi_{-\frac{1}{2}}^+ |1\rangle = 0), \\ \langle 0| \psi_{\frac{1}{2}}^- &= 0 \quad (\psi_{-\frac{1}{2}}^- |0\rangle = 0). \end{aligned} \quad (5.12)$$

Note also that for an odd parameter λ ,

$$\langle m| \lambda = \begin{cases} \lambda \langle m| & \text{for } m \geq 1, \\ -\lambda \langle m| & \text{for } m \leq 0, \end{cases} \quad (5.13)$$

which follows from our convention (4.36) for the state $\langle 1|$ and from (5.10).

5.2 Vertex operators

We are now in a position to construct the vertex operators which are defined as

$$\begin{aligned}
V_C(\zeta; x, \xi, t^\pm) \langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) &= \langle 0 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) C(\zeta) \\
&= \langle 0 | \{ c(z) + \gamma(z)(\partial_{\xi'} + \xi' \partial_{x'}) \} e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) A_M(x', \xi', t^\pm; \zeta) \Big|_{x'=x, \xi'=\xi}, \\
V_B(\zeta; x, \xi, t^\pm) \langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) &= \langle 2 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) B(\zeta) \\
&= \langle 2 | \{ \beta(z) + b(z)(\partial_{\xi'} + \xi' \partial_{x'}) \} e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) \tilde{A}_M(x', \xi', t^\pm; \zeta) \Big|_{x'=x, \xi'=\xi}, \quad (5.14)
\end{aligned}$$

where $\widehat{\Phi}$, A_M and \tilde{A}_M are given by (4.51), (3.55) and (3.68), respectively. Given these operators, we can express the wave function and its dual in terms of τ function as

$$\begin{aligned}
w_M(x, \xi, t^\pm; \zeta) &= \frac{1}{\tau_M(x, \xi, t^\pm; \zeta)} V_C \tau_M(x, \xi, t^\pm; \zeta), \\
\tilde{w}_M(x, \xi, t^\pm; \zeta) &= \frac{1}{\tau_M(x, \xi, t^\pm; \zeta)} V_B \tau_M(x, \xi, t^\pm; \zeta). \quad (5.15)
\end{aligned}$$

In order to obtain the above vertex operators, it is sufficient to find the operators

$$\begin{aligned}
v_c(z; x, \xi, t^\pm) \langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) &= \langle 0 | c(z) e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm), \\
v_\gamma(z; x, \xi, t^\pm) \langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) &= \langle 0 | \gamma(z) e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm), \\
v_b(z; x, \xi, t^\pm) \langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) &= \langle 2 | b(z) e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm), \\
v_\beta(z; x, \xi, t^\pm) \langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm) &= \langle 2 | \beta(z) e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^\pm). \quad (5.16)
\end{aligned}$$

V_C and V_B are then given by

$$\begin{aligned}
V_C(\zeta; x, \xi, t^\pm) &= A_M v_c(z; x, \xi, t^\pm) - (DA_M) v_\gamma(z; x, \xi, t^\pm), \\
V_B(\zeta; x, \xi, t^\pm) &= \tilde{A}_M v_\beta(z; x, \xi, t^\pm) - (D\tilde{A}_M) v_b(z; x, \xi, t^\pm). \quad (5.17)
\end{aligned}$$

For simplicity, let us consider the case $x = \xi = 0$ and define

$$\begin{aligned}
v_c(z; t^\pm) \langle 1 | \widehat{\Phi}(t^\pm) &= \langle 0 | c(z) \widehat{\Phi}(t^\pm), & v_b(z; t^\pm) \langle 1 | \widehat{\Phi}(t^\pm) &= \langle 2 | b(z) \widehat{\Phi}(t^\pm), \\
v_\gamma(z; t^\pm) \langle 1 | \widehat{\Phi}(t^\pm) &= \langle 0 | \gamma(z) \widehat{\Phi}(t^\pm), & v_\beta(z; t^\pm) \langle 1 | \widehat{\Phi}(t^\pm) &= \langle 2 | \beta(z) \widehat{\Phi}(t^\pm). \quad (5.18)
\end{aligned}$$

Provided we can find these operators, the expressions for the general case with $x \neq 0$ and $\xi \neq 0$, can be found using the identity

$$\begin{aligned}
e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t_o^+(\bullet), t_o^-(\bullet), t_e^+(\bullet), t_e^-(\bullet)) \\
= \widehat{\Phi}(t_\xi^+(\bullet), t_\xi^-(\bullet), t_e^+(\bullet) + x \bullet + \xi \bullet \bar{t}_o(\bullet), t_e^-(\bullet) + x \bullet + \xi \bullet \bar{t}_o(\bullet)), \quad (5.19)
\end{aligned}$$

where $\widehat{\Phi}$ is considered as a functional of $t_{o,e}^{\pm}(z)$ and \bullet represents a dummy variable. Now putting the superbosonization formulas for c , γ , b and β , and noting

$$\begin{aligned}\langle 2| e^{-\phi_0^+} z^{-j_0^+} e^{-\phi_{<}^+(z)} &= \langle 1| \gamma_{\frac{1}{2}} e^{-\phi_0^+} z^{-j_0^+} e^{-\phi_{<}^+(z)} \\ &= z^{-1} \langle 0| \gamma_{\frac{1}{2}} = z^{-1} \langle 1| \psi_{\frac{1}{2}}^-, \\ \langle 2| e^{-\phi_0^+} z^{-j_0^+} e^{-\phi_{<}^+(z)} \psi^+(z) &= z^{-1} \langle 1| \psi_{\frac{1}{2}}^- \psi^+(z) = z^{-1} \langle 1| \left(\{ \psi_{\frac{1}{2}}^-, \psi_{-\frac{1}{2}}^+ \} + \psi_{\frac{1}{2}}^- \psi_{>}^+(z) \right) \\ &= z^{-1} \langle 1| \{ 1 + \psi_{\frac{1}{2}}^- \psi_{>}^+(z) \},\end{aligned}$$

we have

$$\begin{aligned}\langle 0| c(z) \widehat{\Phi}(t^{\pm}) &= \langle 1| e^{\phi_{>}^+(z)} \widehat{\Phi}(t^{\pm}), \\ \langle 0| \gamma(z) \widehat{\Phi}(t^{\pm}) &= z \langle 1| \psi_{>}^-(z) e^{\phi_{>}^+(z)} \widehat{\Phi}(t^{\pm}),\end{aligned}\tag{5.20}$$

$$\begin{aligned}\langle 2| b(z) \widehat{\Phi}(t^{\pm}) &= z^{-1} \langle 1| \psi_{\frac{1}{2}}^- e^{-\phi_{>}^+(z)} \widehat{\Phi}(t^{\pm}), \\ \langle 2| \beta(z) \widehat{\Phi}(t^{\pm}) &= \langle 1| \{ 1 + \psi_{\frac{1}{2}}^- \psi_{>}^+(z) \} e^{-\phi_{>}^+(z)} \widehat{\Phi}(t^{\pm}),\end{aligned}\tag{5.21}$$

where

$$\psi_{>}^+(z) = \sum_{n \geq 0} \psi_{n+\frac{1}{2}}^+ z^{-n-1} \quad \text{and} \quad \psi_{>}^-(z) = \sum_{n \geq 0} \psi_{n+\frac{1}{2}}^- z^{-n-2}.\tag{5.22}$$

Next task is to see the effect of $e^{\pm\phi_{>}^+(z)}$ on $\widehat{\Phi}(t^{\pm})$. Here we note that for a formal power series $\tau(w) = \sum_{n \geq 0} \tau_{n+\frac{1}{2}} w^n$ with odd Grassmann parity,

$$\begin{aligned}e^{\phi_{>}^+(z)} \psi^+[\tau(\bullet)] e^{-\phi_{>}^+(z)} &= \oint_{|w| < |z|} dw \tau(w) \left(1 - \frac{w}{z}\right) \psi^+(w) \\ &= \psi^+[\tau(\bullet)], \quad \text{with} \quad \tau(w) \equiv \left(1 - \frac{w}{z}\right) \tau(w) = \sum_{n=0} \left(\tau_{n+\frac{1}{2}} - \frac{1}{z} \tau_{n-\frac{1}{2}}\right) w^n, \\ e^{-\phi_{>}^+(z)} \psi^+[\tau(\bullet)] e^{\phi_{>}^+(z)} &= \oint_{|w| < |z|} dw \tau(w) \frac{1}{1 - \frac{w}{z}} \psi^+(w) \\ &= \psi^+[\tau(\bullet)], \quad \text{with} \quad \tau(w) \equiv \frac{1}{1 - \frac{w}{z}} \tau(w) = \sum_{n=0} \left(\sum_{m=0}^n \tau_{n-m+\frac{1}{2}} z^{-m}\right) w^n,\end{aligned}\tag{5.23}$$

and similarly,

$$\begin{aligned}e^{\phi_{>}^+(z)} \psi^-[\tau(\bullet)] e^{-\phi_{>}^+(z)} &= \oint_{|w| < |z|} dw w \tau(w) \frac{1}{1 - \frac{w}{z}} \psi^-(w) = \psi^-[\tau(\bullet)], \\ e^{-\phi_{>}^+(z)} \psi^-[\tau(\bullet)] e^{\phi_{>}^+(z)} &= \oint_{|w| < |z|} dw w \tau(w) \left(1 - \frac{w}{z}\right) \psi^-(w) = \psi^-[\tau(\bullet)].\end{aligned}\tag{5.24}$$

Moreover we observe for a formal power series $x(w) = \sum_{n=1} x_n w^n$,

$$e^{\pm\phi_{>}^+(z)} e^{j^+[\tau(\bullet)]} = e^{j^+[\tau(\bullet)]},\tag{5.25}$$

where

$$j^+_{[z]_{\pm}} x(\bullet) = \sum_{n=1} j_n^+ (x_n \mp \frac{z^{-n}}{n}), \quad (5.26)$$

and hence

$$_{[z]_{\pm}} x(w) \equiv x(w) \pm \log(1 - \frac{w}{z}). \quad (5.27)$$

Suppose that $f(\tau(\bullet))$ be a functional of $\tau(z)$ and imagine $\mathcal{T}(z; \tau)$ be a differential operator with respect to $\{\tau_{n+\frac{1}{2}}\}$ such that

$$e^{\mathcal{T}(z; \tau)} f(\tau(\bullet)) = f(_{[z]_+} \tau(\bullet)). \quad (5.28)$$

Clearly, the inverse operation gives

$$e^{-\mathcal{T}(z; \tau)} f(\tau(\bullet)) = f(_{[z]_-} \tau(\bullet)). \quad (5.29)$$

One can see such an operator is given by

$$\mathcal{T}(z; \tau) = \sum_{n=0} \sum_{m=1} \frac{-1}{m} z^{-m} \tau_{n+\frac{1}{2}} \partial_{\tau_{n+m+\frac{1}{2}}}. \quad (5.30)$$

Therefore we have

$$\begin{aligned} e^{\pm \phi_{>}^+(z)} \widehat{\Phi}(t^{\pm}(\bullet)) &= \widehat{\Phi}(_{[z]_{\pm}} t_o^+(\bullet), _{[z]_{\mp}} t_o^-(\bullet), _{[z]_{\pm}} t_e^+(\bullet), t_e^-(\bullet)) \\ &= e^{\pm \{ \sum_{n=1} \frac{-1}{n} z^{-n} \partial_{t_{2n}^+} + \mathcal{T}(z; t_o^+) - \mathcal{T}(z; t_o^-) \}} \widehat{\Phi}(t^{\pm}(\bullet)) \\ &= e^{\pm \sum_{n=1} \frac{-1}{n} z^{-n} D_{2n}^+} \widehat{\Phi}(t^{\pm}(\bullet)) \end{aligned} \quad (5.31)$$

where D_{2n}^+ is the operator defined previously in (3.33). This is expected because from

$$D_{2n+1}^{\pm} \widehat{\Phi}(t^{\pm}) = \pm \psi_{n+\frac{1}{2}}^{\pm} \widehat{\Phi}(t^{\pm}) \quad (5.32)$$

and the fact that D_n^{\pm} , $n = 1, 2, \dots$ obeys the same algebra of the annihilation part of $\mathbf{A}_{j\psi}$ when $[\ , \]_{\pm}$ is replaced by $-[\ , \]_{\pm}$, with the correspondence

$$D_{2n+1}^{\pm} \leftrightarrow \pm \psi_{n+\frac{1}{2}}^{\pm}, \quad D_{2n}^{\pm} \leftrightarrow j_n^{\pm}.$$

D_{2n}^{\pm} must give the same effect on $\widehat{\Phi}(t^{\pm})$ as j_n^{\pm} does. Actually, one can observe for example

$$\begin{aligned} e^{\pm \sum \frac{-1}{n} z^{-n} D_{2n}^+} D_{2m+1}^{\pm} e^{\mp \sum \frac{-1}{n} z^{-n} D_{2n}^+} &= \sum_{n=0} z^{-n} D_{2(m+n)+1}^{\pm}, \\ e^{\pm \sum \frac{-1}{n} z^{-n} D_{2n}^+} D_{2m+1}^{\mp} e^{\mp \sum \frac{-1}{n} z^{-n} D_{2n}^+} &= D_{2m+1}^{\mp} - \frac{1}{w} D_{2(m+n)+1}^{\mp}, \end{aligned} \quad (5.33)$$

which are equivalent to (5.23) and (5.24) when making replacement $D_{2n+1}^{\pm} \rightarrow \pm \psi_{n+\frac{1}{2}}^{\pm}$ and $-D_{2n}^{\pm} \rightarrow j_n^{\pm}$. Now from (5.20), (5.21), (5.31) and (5.32), we obtain

$$\begin{aligned}
v_c(z; t^{\pm}) &= e^{\sum \frac{-1}{n} z^{-n} D_{2n}^+}, \\
v_{\gamma}(z; t^{\pm}) &= e^{\sum \frac{-1}{n} z^{-n} D_{2n}^+} \sum_{m=0} -z^{-(m+1)} D_{2m+1}^- \\
&= -z^{-1} D_1^- e^{\sum \frac{-1}{n} z^{-n} D_{2n}^+}, \\
v_b(z; t^{\pm}) &= e^{-\sum \frac{-1}{n} z^{-n} D_{2n}^+} (-z^{-1} D_1^-), \\
v_{\beta}(z; t^{\pm}) &= e^{-\sum \frac{-1}{n} z^{-n} D_{2n}^+} (1 + \sum_{m=0} z^{-(m+1)} D_{2m+1}^+ D_1^-) \\
&= e^{-\sum \frac{-1}{n} z^{-n} D_{2n}^+} + z^{-1} D_1^+ e^{-\sum \frac{-1}{n} z^{-n} D_{2n}^+} D_1^-. \tag{5.34}
\end{aligned}$$

Moreover we can find the expressions for $x \neq 0$ and $\xi \neq 0$, by noting

$$\begin{aligned}
v_{\alpha}(z; x, \xi, t^{\pm}) &\langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^{\pm}) \\
&= e^{\xi(D_1^+ - D_1^-)} v_{\alpha}(z; t^{\pm}) e^{-\xi(D_1^+ - D_1^-)} \langle 1 | e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^{\pm}) \tag{5.35}
\end{aligned}$$

where the subscript α stands for c, γ, b and β . Using the above formula and taking into account[¶]

$$D e^{xj_1^t + \xi\sigma} = \sigma e^{xj_1^t + \xi\sigma} \tag{5.36}$$

and

$$\langle 1 | \sigma = \langle 1 | (\psi_{\frac{1}{2}}^+ + \psi_{\frac{1}{2}}^-) = \langle 1 | \psi_{\frac{1}{2}}^-,$$

we obtain consequently

$$\begin{aligned}
v_c(z; x, \xi, t^{\pm}) &= e^{\sum \frac{-1}{n} z^{-n} \{ D_{2n}^+ + \xi(D_{2n+1}^+ + D_{2n+1}^-) \}}, \\
v_{\gamma}(z; x, \xi, t^{\pm}) &= z^{-1} D e^{\sum \frac{-1}{n} z^{-n} \{ D_{2n}^+ + \xi(D_{2n+1}^+ + D_{2n+1}^-) \}}, \\
v_b(z; x, \xi, t^{\pm}) &= z^{-1} e^{-\sum \frac{-1}{n} z^{-n} \{ D_{2n}^+ + \xi(D_{2n+1}^+ + D_{2n+1}^-) \}} D, \\
v_{\beta}(z; x, \xi, t^{\pm}) &= e^{-\sum \frac{-1}{n} z^{-n} \{ D_{2n}^+ + \xi(D_{2n+1}^+ + D_{2n+1}^-) \}} \\
&\quad - z^{-1} \{ D_1^+ + \xi \partial_x \} e^{-\sum \frac{-1}{n} z^{-n} \{ D_{2n}^+ + \xi(D_{2n+1}^+ + D_{2n+1}^-) \}} D. \tag{5.37}
\end{aligned}$$

Let us reflect on the results from the view point of the superbosonization. We have first realized the negative frequency part of $\widehat{gl|1}$ as differential operators which act on the polynomial ring $\mathcal{B} = \mathbf{C}[[t_m^+, t_m^-; m \geq 1]]$. Then, based on the Kac-van de Leur superbosonization, we have obtained the description of the B-C system on \mathcal{B} . (More

[¶] The fact that

$$e^{\xi(D_1^+ - D_1^-)} e^{-\xi D} = e^{\xi \sum \bar{t}_{2m+1} \partial_{t_{2(m+1)}}} e^{\xi(\partial_{t_1^+} - \partial_{t_1^-} - \partial_{\xi})}$$

acts as an identity on $e^{xj_1^t + \xi\sigma} \widehat{\Phi}(t^{\pm})$ implies the relation (5.19).

precisely, we must introduce one more variable which counts the total charge.) Consider the vertex operators

$$\begin{aligned} V_c(z; t^\pm) \langle 1 | \hat{\Phi}(t^\pm) &= \langle 0 | \hat{\Phi}(t^\pm) c(z), & V_b(z; t^\pm) \langle 1 | \hat{\Phi}(t^\pm) &= \langle 2 | \hat{\Phi}(t^\pm) b(z), \\ V_\gamma(z; t^\pm) \langle 1 | \hat{\Phi}(t^\pm) &= \langle 0 | \hat{\Phi}(t^\pm) \gamma(z), & V_\beta(z; t^\pm) \langle 1 | \hat{\Phi}(t^\pm) &= \langle 2 | \hat{\Phi}(t^\pm) \beta(z). \end{aligned} \quad (5.38)$$

From (5.14) and (5.14) and the expressions of A_M and \tilde{A}_M , these are given by

$$\begin{aligned} V_c(z; t^\pm) &= e^{t_e^+(z) - \frac{1}{2} z t_o^+ t_o^-(z)} \{ v_c(z; t^\pm) + z t_o^-(z) v_\gamma(z; t^\pm) \}, \\ V_\gamma(z; t^\pm) &= e^{t_e^-(z) + \frac{1}{2} z t_o^+ t_o^-(z)} \{ v_\gamma(z; t^\pm) + t_o^+(z) v_c(z; t^\pm) \}, \\ V_b(z; t^\pm) &= e^{-t_e^+(z) - \frac{1}{2} z t_o^+ t_o^-(z)} \{ v_b(z; t^\pm) + t_o^+(z) v_\beta(z; t^\pm) \}, \\ V_\beta(z; t^\pm) &= e^{-t_e^-(z) + \frac{1}{2} z t_o^+ t_o^-(z)} \{ v_\beta(z; t^\pm) - z t_o^-(z) v_b(z; t^\pm) \}, \end{aligned} \quad (5.39)$$

which may be viewed as a kind of superbosonization realized on \mathcal{B} . The expression of the bilinear identity for the τ -function (super-Hirota equation) can be obtained without any difficulty although we have not written down here.

6 Concluding Remarks

We have established the operator theory for the SKP hierarchies and formulated the maximal SKP hierarchy which includes all the flows of the known SKP hierarchies in a unified way and allows the τ -function description of the theory. Now the relation between different SKP hierarchies is very clear.

Here let us look briefly at the geometrical aspects of the subject, which we have not touched on so far. As stated in Ref.[2], for a general B-C system, the fields B and C are defined on an arbitrary (1|1) dimensional complex supermanifold (i.e., a super curve in general without any superconformal structure), and are considered as sections of line bundles ω^k and ω^{1-k} (in our setting, $k = 1$) respectively. According to the coordinate transformation $(z, \theta) \rightarrow (\tilde{z}, \tilde{\theta})$, a section $\sigma \in \omega$ transforms as $\sigma = \tilde{\sigma} \frac{\partial(\tilde{z}, \tilde{\theta})}{\partial(z, \theta)}$. It should be noted here that the moduli space of supercurves is coincident with that of the $N = 2$ super-Riemann surface[21]. The hidden $N = 2$ superconformal symmetry of the B-C system originates in this coincidence. In Ref.[22] the super-Krichever construction was studied and the geometrical meaning of the SKP hierarchies was clarified. Their arguments ought to be traced in the field theoretic context. The super-Krichever map assigns injectively a point of the super-Grassmannian, hence a state in the B-C Fock space, to a set of geometrical data of an arbitrary supercurve and a line bundle ω^k on it. The same map was also investigated in constructing the operator formalism for the superstring theory[19]. From these studies it becomes clear that as long as we work within the frame of our restricted super-Grassmannian Gr_0 or equivalently the space of the wave operators Γ_0 , the supercurves we can cover are constrained excluding super-Riemann surfaces of genus $g \neq 1$. This is a disappointing fact. However if we convert

the description of the theory in terms of the τ -function, we would be able to generalize the theory to cover more general super-Grassmannians. In this relation, comparison with the τ -function of A.S. Schwarz[26] which is defined in more abstract and general manner would be helpful. An interesting problem is to find, at least in the genus one case, some characterization of the locus of the geometrical data coming from $N = 1$ super-Riemann surfaces inside the larger moduli space of the geometrical data coming from general supercurves. To think over these issues, it would be useful to recall another SKP hierarchy given by LeClair[20]. It is based on the superbosonization which preserves the superconformal symmetry manifestly[23]. In this superbosonization the B-C system is expressed in terms of the following currents:

$$J = -BC, \quad J^* = -D \log B, \\ J(z_1, \theta_1)J^*(z_2, \theta_2) \sim \frac{1}{z_1 - z_2 - \theta_1\theta_2}, \quad J(1)J(2) \sim \text{regular}, \quad J^*(1)J^*(2) \sim \text{regular}.$$

The τ -function of this SKP hierarchy is defined as

$$\tau(t, t^*) = \langle 1 | e^{J^*[t^*(\bullet)] + J[t(\bullet)]} | \Xi_{0<} \rangle,$$

where $J[t(\bullet)]$ stands for the expression (4.60) and $J^*[t^*(\bullet)]$ is defined similarly. From the above expression we recognize that this SKP hierarchy is essentially JSKP hierarchy, since the former reduces to the latter by setting $t_n^* = 0$, $n = 1, 2, \dots$. On the other hand we can see that the state $e^{J^*[t^*(\bullet)]} | \Xi_{0<} \rangle$ does not satisfy the bilinear identity. The operator $e^{J^*[t^*(\bullet)]}$ cannot be considered as the time evolution operator on the super-Grassmannian and is introduced in order to keep the whole information of the state $| \Xi_{0<} \rangle$. From the speciality of this superbosonization, when the initial state $| \Xi_{0<} \rangle$ is associated with a super-Riemann surface, its superconformal structure is preserved and translated into the one expressed in terms of the time variables t_n and t_n^* .

In addition to the geometrical consideration, there are several directions for future work. For the KP hierarchy N soliton solutions are obtained by successively applying the vertex operators to the vacuum state. The same procedure would be also applicable for the SKP hierarchies. The operator theory for the SBKP hierarchy and its τ -function description has already been studied in [25] (see also [6]). Making similar efforts as we did in the previous section, we will be able to transfer their elaborated result to the SBKP version of our maximal SKP hierarchy, which will be more accessible. Though it is not clear at present that the SKP hierarchies have relevance to physics such as the two-dimensional supergravity, it would be interesting on its own to seek additional symmetries, the relation to super- W algebras and favorable reductions to the KdV type together with their τ -function description.

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A Calculation of $\langle 2 | B_j | X_{<} \rangle$

Let us show

$$\frac{1}{\langle 1 | X_{<} \rangle} \langle 2 | B_j | X_{<} \rangle = \frac{(-1)^{j+1}}{\text{sdet}_\pi X_{<<}} \text{sdet}_\pi X_{<<}^{(j)}, \quad j = -1, 0, 1, \dots \quad (\text{A.1})$$

Using the formula (4.26) we have

$$\langle 2 | B_j | X_{<} \rangle = \int \prod_{r>0} d\tilde{\gamma}_r \int \prod_{l \geq 1} d\tilde{c}_l \tilde{\gamma}_{\frac{1}{2}} (-1)^j \partial_{\tilde{C}_{-j}} \prod_{k \geq 1} \tilde{c}(X)_k \prod_{s>0} \delta(\tilde{\gamma}(X)_s) \Big|_{\tilde{C}_{<}=0}. \quad (\text{A.2})$$

Then we change the integration variables $\tilde{C}_j \rightarrow \tilde{C}'_j = \tilde{C}(X)_j$ as in (4.33), provided that in this case $j = 1, 2, \dots$ and \tilde{C}_{\leq} are considered as auxiliary variables. The right-hand side of the above expression becomes

$$(-1)^j \text{sdet} X_{<<} \int \prod_{r>0} d\tilde{\gamma}'_r \int \prod_{l \geq 1} d\tilde{c}'_l \sum_{k>0} (\tilde{C}'_k - \sum_{l \geq 0} \tilde{C}_{-l} X_{l,-k}) (X_{<<}^{-1})_{-k,-1} \sum_{m>0} X_{j,-m} \partial_{\tilde{C}'_m} \prod_{k \geq 1} \tilde{c}'_k \prod_{s>0} \delta(\tilde{\gamma}'_s) \Big|_{\tilde{C}_{<}=0}. \quad (\text{A.3})$$

Integrating by part we have

$$= (-1)^{j+1} \text{sdet} X_{<<} \cdot \sum_{k>0} X_{j,-k} (X_{<<}^{-1})_{-k,-1}. \quad (\text{A.4})$$

Since $\langle 1 | X_{<} \rangle = \text{sdet} X_{<<}$, we thus obtain

$$\frac{1}{\langle 1 | X_{<} \rangle} \langle 2 | B_j | X_{<} \rangle = (-1)^{j+1} \sum_{k>0} X_{j,-k} (X_{<<}^{-1})_{-k,-1}. \quad (\text{A.5})$$

It is not hard to see that

$$\frac{1}{\text{sdet}_\pi X_{<<}} \text{sdet}_\pi X_{<<}^{(j)} = \sum_{k>0} X_{j,-k} (X_{<<}^{-1})_{-k,-1}, \quad (\text{A.6})$$

which is viewed as a co-factor expansion of the superdeterminant $\text{sdet}_\pi X_{<<}^{(j)}$. Taking into account

$$(X_{<<}^{-1})^{01} = -X_{<<}^{00}{}^{-1} X_{<<}^{01} \widehat{X}_{<<}^{11}{}^{-1} \quad \text{and} \quad (X_{<<}^{-1})^{11} = \widehat{X}_{<<}^{11}{}^{-1} \quad (\text{A.7})$$

where the notation (3.40) is used, we can reduce (A.6) to an exercise for an ordinary matrix determinant.

B More about the superbosonization

As mentioned in §5, in the β - γ system, local fields which carry the picture charge exist and are most easily identified in the FMS bosonization[18]. These are

$$\delta(\beta(z)) = e^{\phi_0^-} z^{-j_0^-} e^{\phi_<^-(z)} e^{\phi_>^-(z)}, \quad \delta(\gamma(z)) = e^{-\phi_0^-} z^{j_0^-} e^{-\phi_<^-(z)} e^{-\phi_>^-(z)}, \quad (\text{B.1})$$

and the additional free fermionic pair independent of $\delta(\beta(z))$ and $\delta(\gamma(z))$,

$$\xi(z) = \Theta(\beta(z)) \quad \text{and} \quad \eta(z) = \partial_z \Theta(\gamma(z)), \quad (\text{B.2})$$

where Θ is Heviside step function[27]. FMS bosonization says the original β , γ fields are reconstructed from the above fields as

$$\beta(z) = \partial_z \Theta(\beta(z)) \cdot \delta(\gamma(z)), \quad \gamma(z) = \partial_z \Theta(\gamma(z)) \cdot \delta(\beta(z)). \quad (\text{B.3})$$

Obviously the formulas (B.1) are considered to be the same as in the superbosonization we discussed. With this in mind we can find from (B.2) or (B.3) the superbosonization rules for ξ and η fields:

$$\begin{aligned} \eta(z) &= \lim_{w \rightarrow z} \psi^-(z) c(w) \delta(\gamma(w)) \\ &= e^{\phi_0^+ - \phi_0^-} e^{\phi_<^+(z) - \phi_<^-(z)} \psi^-(z) z^{j_0^+ + j_0^-} e^{\phi_>^+(z) - \phi_>^-(z)}, \\ \partial_z \xi(z) &= \lim_{w \rightarrow z} \psi^+(z) b(w) \delta(\beta(w)) \\ &= e^{-\phi_0^+ + \phi_0^-} e^{-\phi_<^+(z) + \phi_<^-(z)} \psi^+(z) z^{-(j_0^+ + j_0^-)} e^{-\phi_>^+(z) + \phi_>^-(z)}. \end{aligned} \quad (\text{B.4})$$

These expressions already appeared in [24]

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